Lusin Area Function and Molecular Characterizations of Musielak-Orlicz Hardy Spaces and Their Applications

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Abstract Let $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ be a function such that $\varphi(x, \cdot)$ is an Orlicz function, and $\varphi(\cdot, t)$ is a Muckenhoupt $A_\infty(\mathbb{R}^n)$ weight. In this paper, the authors establish the Lusin area function and the molecular characterizations of the Musielak-Orlicz Hardy space $H_\varphi(\mathbb{R}^n)$ introduced by Luong Dang Ky via the grand maximal function. As an application, the authors obtain the φ-Carleson measure characterization of the Musielak-Orlicz BMO-type space $\mathrm{BMO}_\varphi(\mathbb{R}^n)$, which was proved to be the dual space of $H_\varphi(\mathbb{R}^n)$ by Luong Dang Ky.

1 Introduction

The real-variable theory of Hardy spaces on the n-dimensional Euclidean space $\mathbb{R}^n$ was originally studied by Stein and Weiss [50] and systematically developed by Fefferman and Stein in a seminal paper [19]. Since the Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ is, especially when studying the boundedness of operators, a suitable substitute of the Lebesgue space $L^p(\mathbb{R}^n)$, it plays an important role in various fields of analysis and partial differential equations (see, for example, [13, 47, 49] and their references). In order to conveniently apply the real-variable theory of $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$, their several equivalent characterizations were revealed one after the other (see, for example, [19, 12, 37, 53, 49]). Among others a very important and useful characterization of the Hardy spaces is their atomic characterizations, which were obtained by Coifman [12] when $n = 1$ and Latter [37] when $n > 1$. Later, as an extension of this characterization, the molecular characterization of Hardy spaces was established by Taibleson and Weiss [53].

On the other hand, due to need for more inclusive classes of function spaces than the $L^p(\mathbb{R}^n)$-families from applications, the Orlicz space was introduced by Birnbaum-Orlicz in

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[3] and Orlicz in [43], which is widely used in various branches of analysis (see, for example, [44, 45, 1, 9, 26, 27, 55, 29, 30, 31] and their references). Moreover, as a development of the theory of Orlicz spaces, Orlicz-Hardy spaces and their dual spaces were studied by Strömberg [51] and Janson [28] on $\mathbb{R}^n$ and Viviani [54] on spaces of homogeneous type in the sense of Coifman and Weiss [15] and, quite recently, Orlicz-Hardy spaces associated with operators by Jiang and Yang [29, 30], Jiang, Yang and Zhou [31].

Furthermore, the classical BMO space (the *space of functions with bounded mean oscillation*), originally introduced by John and Nirenberg [32], and the classical Morrey space, originally by Morrey [40], play an important role in the study of partial differential equations and harmonic analysis (see, for example, [19, 42] for further details). In particular, Fefferman and Stein [19] proved that $\mathrm{BMO}(\mathbb{R}^n)$ is the dual space of $H^1(\mathbb{R}^n)$ and also obtained the Carleson measure characterization of $\mathrm{BMO}(\mathbb{R}^n)$. Recall that the Carleson measure was originally introduced by Carleson [10, 11] and its equivalent characterization, in terms of $\mathrm{BMO}(\mathbb{R}^n)$ function, was established in [10]. Moreover, the generalized BMO-type space $\mathrm{BMO}_\rho(\mathbb{R}^n)$ was studied in [51, 28, 54, 25] and it was proved therein to be the dual space of the Orlicz-Hardy space $H_\Phi(\mathbb{R}^n)$, where Φ denotes the Orlicz function on $(0,\infty)$ and $\rho(t) := t^{-1}/\Phi^{-1}(t^{-1})$ for all $t \in (0,\infty)$. Here and in what follows, Φ^{-1} denotes the *inverse function* of Φ. Meanwhile, the Carleson measure characterization of $\mathrm{BMO}_\rho(\mathbb{R}^n)$ was obtained in [25].

Recently, a new Musielak-Orlicz Hardy space $H_\varphi(\mathbb{R}^n)$ was introduced by Ky [34], via the grand maximal function, which includes both the Orlicz-Hardy space in [51, 28] and the weighted Hardy space $H^p_\omega(\mathbb{R}^n)$ with $p \in (0,1]$ and $\omega \in A_\infty(\mathbb{R}^n)$ in [23, 52]. Here and in what follows, $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is a function such that $\varphi(x,\cdot)$ is an Orlicz function of uniformly upper type 1 and lower type p for some $p \in (0,1]$ (see Section 2 for the definitions of uniformly upper or lower types), and $\varphi(\cdot,t)$ is a Muckenhoupt $A_\infty(\mathbb{R}^n)$ weight, and $A_q(\mathbb{R}^n)$ with $q \in [1,\infty]$ denotes the *class of Muckenhoupt's weights* (see, for example, [22, 23, 24] for their definitions and properties). In [34], Ky first established the atomic characterization of $H_\varphi(\mathbb{R}^n)$, and further introduced the Musielak-Orlicz BMO-type space $\mathrm{BMO}_\varphi(\mathbb{R}^n)$ and proved that it is the dual space of $H_\varphi(\mathbb{R}^n)$. Furthermore, some interesting applications of these spaces were also presented in [4, 6, 7, 34, 35, 36]. Moreover, the local Musielak-Orlicz Hardy space, $h_\varphi(\mathbb{R}^n)$, and its dual space, $\mathrm{bmo}_\varphi(\mathbb{R}^n)$, were studied in [56] and some applications of $h_\varphi(\mathbb{R}^n)$ and $\mathrm{bmo}_\varphi(\mathbb{R}^n)$, to pointwise multipliers of BMO-type spaces and to the boundedness of local Riesz transforms and pseudo-differential operators on $h_\varphi(\mathbb{R}^n)$, were also obtained in [56]. Recall that Musielak-Orlicz functions are the natural generalization of Orlicz functions that may vary in the spatial variables (see, for example, [17, 18, 34, 41]). Moreover, the motivation to study function spaces of Musielak-Orlicz type is due to that they have wide applications to several branches of physics and mathematics (see, for example, [5, 6, 7, 17, 18, 34, 38, 56] for more details).

Motivated by [34, 53, 19, 10], in this paper, we establish the Lusin area function and the molecular characterizations of the Musielak-Orlicz Hardy space $H_\varphi(\mathbb{R}^n)$. As an application, we obtain the φ-Carleson measure characterization of the Musielak-Orlicz BMO-type space $\mathrm{BMO}_\varphi(\mathbb{R}^n)$.

Precisely, this paper is organized as follows. In Section 2, we recall some notions of growth functions, some examples, and their properties established in [34].

In Section 3, we first recall some notions about tent spaces and then study the Musielak-Orlicz tent space $T_\varphi(\mathbb{R}^{n+1}_+)$. The main target of this section is to establish the atomic characterization of $T_\varphi(\mathbb{R}^{n+1}_+)$ (see Theorem 3.1 below). As a byproduct, we show that if $f \in T_\varphi(\mathbb{R}^{n+1}_+) \cap T^p_2(\mathbb{R}^{n+1}_+)$ with $p \in (0, \infty)$, then the atomic decomposition of f holds in both $T_\varphi(\mathbb{R}^{n+1}_+)$ and $T^p_2(\mathbb{R}^{n+1}_+)$, which plays an important role in the remainder of this paper (see Corollary 3.4 below).

In Section 4, we introduce the Hardy-type spaces, $H_{\varphi,S}(\mathbb{R}^n)$ and $H^{q,s,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$, respectively, via the Lusin area function and the molecule, and then prove that the operator π_ϕ, which was first introduced in [14] (see also (4.2) below), maps the Musielak-Orlicz tent space $T_\varphi(\mathbb{R}^{n+1}_+)$ continuously into $H_{\varphi,S}(\mathbb{R}^n)$ (see Proposition 4.7 below). By this and the atomic decomposition of $T_\varphi(\mathbb{R}^{n+1}_+)$, we conclude that for each $f \in H_{\varphi,S}(\mathbb{R}^n)$ vanishing weakly at infinity (see Section 4 below for the definition of vanishing weakly at infinity), there exists a molecular decomposition of f holding in $H_{\varphi,S}(\mathbb{R}^n)$ (see Proposition 4.8 below). Via this molecular decomposition of $H_{\varphi,S}(\mathbb{R}^n)$ and the atomic characterization of $H_\varphi(\mathbb{R}^n)$ established by Ky [34], we further obtain the Lusin area function and the molecular characterizations of $H_\varphi(\mathbb{R}^n)$ (see Theorem 4.11 below).

In Section 5, we first recall the definition of the Musielak-Orlicz BMO-type space $\mathrm{BMO}_\varphi(\mathbb{R}^n)$ and introduce the φ-Carleson measure. When φ further satisfies $nq(\varphi) < (n+1)i(\varphi)$ (see (2.3) and (2.4) below respectively for the definitions of $q(\varphi)$ and $i(\varphi)$), then in Theorem 5.3 below, we establish the φ-Carleson measure characterization of $\mathrm{BMO}_\varphi(\mathbb{R}^n)$ by using the Lusin area function characterization of $H_\varphi(\mathbb{R}^n)$ in Theorem 4.11.

We remark that the method obtaining the Lusin area function characterization of $H_\varphi(\mathbb{R}^n)$ in this paper is different from the classical case. More precisely, in the classical case, the Lusin area function characterization of Hardy spaces was established by using the Calderón reproducing formula and a subtle decomposition of all dyadic cubes in $\mathbb{R}^n$ (see, for example, [20]). However, in this paper, we establish the Lusin area function characterization of $H_\varphi(\mathbb{R}^n)$ by using the Calderón reproducing formula (see (4.22) below), the atomic decomposition of the Musielak-Orlicz tent space in Theorem 3.1 and some boundedness of the operator π_ϕ in Proposition 4.7. This method is more close to the method used in [14, 30, 29, 31, 48, 8]. Moreover, different from [48, 8], we do not need the additional assumption that for any $t \in [0, \infty)$, $\varphi(\cdot, t)$ satisfies the reverse Hölder inequality of order 2 (see Definition 2.1 below for the definition of the reverse Hölder inequality), by fully using the $L^p(\mathbb{R}^n)$ boundedness of the Lusin area function S for all $p \in (1, \infty)$. However, in [8], by the assumptions of operator L, it is known that the Lusin area function S_L, associated with the operator L, is bounded *only* on $L^2(\mathcal{X})$. Thus, in some sense, the better properties of S than S_L make up the absence of the reverse Hölder property of weights in this paper.

Moreover, by using the Lusin area function characterization of $H_\varphi(\mathbb{R}^n)$, obtained in this paper, Liang, Huang and Yang [39], via establishing a Musielak-Orlicz Fefferman-Stein vector-valued inequality, further obtain the Littlewood-Paley g-function and g^*_λ-function characterizations of $H_\varphi(\mathbb{R}^n)$, under the *additional assumption* that $\varphi(\cdot, t)$ being a Muckenhoupt $A_2(\mathbb{R}^n)$ weight. Furthermore, the characterizations of $H_\varphi(\mathbb{R}^n)$ in terms of the vertical and the non-tangential maximal functions, with $\varphi(\cdot, t)$ being a Muckenhoupt $A_\infty(\mathbb{R}^n)$ weight, were obtained in [39].

We also point out that the main results of this paper, including the Lusin area function and the molecular characterizations of $H_\varphi(\mathbb{R}^n)$ and the φ-Carleson measure characterization of $\mathrm{BMO}_\varphi(\mathbb{R}^n)$, have local variants, which will be studied in a forthcoming paper (see [56] for the definition of the local Musielak-Orlicz Hardy space $h_\varphi(\mathbb{R}^n)$).

Finally we make some conventions on notation. Throughout the whole paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. We also use $C(\gamma, \beta, \cdots)$ to denote a *positive constant depending on the indicated parameters* γ, β, $\cdots$. The *symbol* $A \lesssim B$ means that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. The *symbol* $\lfloor s \rfloor$ for $s \in \mathbb{R}$ denotes the maximal integer not more than s. For any given normed spaces $\mathcal{A}$ and $\mathcal{B}$ with the corresponding norms $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$, the *symbol* $\mathcal{A} \subset \mathcal{B}$ means that for all $f \in \mathcal{A}$, then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \lesssim \|f\|_{\mathcal{A}}$. For any subset E of $\mathbb{R}^n$, we denote by $E^\complement$ the *set* $\mathbb{R}^n \setminus E$ and by χ_E its *characteristic function*. We also set $\mathbb{N} := \{1, 2, \cdots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. For any $\theta := (\theta_1, \ldots, \theta_n) \in \mathbb{Z}_+^n$, let $|\theta| := \theta_1 + \cdots + \theta_n$ and $\partial_x^\theta := \frac{\partial^{|\theta|}}{\partial x_1^{\theta_1} \cdots \partial x_n^{\theta_n}}$. For any index $q \in [1, \infty]$, we denote by q' its *conjugate index*, namely, $1/q + 1/q' = 1$.

2 Growth functions

In this section, we first recall some notions and assumptions on growth functions considered in this paper and give some examples which satisfy these assumptions. We also recall some properties of growth functions established in [34].

Recall that a function $\Phi : [0, \infty) \to [0, \infty)$ is called an *Orlicz function* if it is nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for $t \in (0, \infty)$ and $\lim_{t\to\infty} \Phi(t) = \infty$ (see, for example, [41, 44, 45]). The function Φ is said to be of *upper type* p (resp. *lower type* p) for some $p \in [0, \infty)$, if there exists a positive constant C such that for all $t \in [1, \infty)$ (resp. $t \in [0, 1]$) and $s \in [0, \infty)$, $\Phi(st) \le Ct^p\Phi(s)$.

For a given function $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ such that for any $x \in \mathbb{R}^n$, $\varphi(x, \cdot)$ is an Orlicz function, φ is called to be of *uniformly upper type* p (resp. *uniformly lower type* p) for some $p \in [0, \infty)$ if there exists a positive constant C such that for all $x \in \mathbb{R}^n$, $t \in [0, \infty)$ and $s \in [1, \infty)$ (resp. $s \in [0, 1]$), $\varphi(x, st) \le Cs^p\varphi(x, t)$. We say that φ is of *positive uniformly upper type* (resp. *uniformly lower type*) if it is of uniformly upper type (resp. uniformly lower type) p for some $p \in (0, \infty)$, and let

$$i(\varphi) := \sup\{p \in (0, \infty) : \ \varphi \text{ is of uniformly lower type } p\}. \tag{2.1}$$

Observe that $i(\varphi)$ may not be attainable, namely, φ may not be of uniformly lower type $i(\varphi)$; see below for some examples.

Let $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ satisfy that $x \mapsto \varphi(x, t)$ is measurable for all $t \in [0, \infty)$. Following [34], $\varphi(\cdot, t)$ is called *uniformly locally integrable* if, for all compact sets K in $\mathbb{R}^n$,

$$\int_K \sup_{t\in(0,\infty)} \left\{ \varphi(x, t) \left[\int_K \varphi(y, t)\, dy \right]^{-1} \right\} dx < \infty.$$

Definition 2.1. Let $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be uniformly locally integrable. The function $\varphi(\cdot,t)$ is called to satisfy the *uniformly Muckenhoupt condition for some* $q \in [1,\infty)$, denoted by $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$, if, when $q \in (1,\infty)$,

$$\mathbb{A}_q(\varphi) := \sup_{t\in[0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|^q} \int_B \varphi(x,t)\,dx \left\{ \int_B [\varphi(y,t)]^{-q'/q}\,dy \right\}^{q/q'} < \infty, \tag{2.2}$$

where $1/q + 1/q' = 1$, or

$$\mathbb{A}_1(\varphi) := \sup_{t\in[0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int_B \varphi(x,t)\,dx \left(\operatorname*{esssup}_{y\in B} [\varphi(y,t)]^{-1} \right) < \infty.$$

Here the first supremums are taken over all $t \in [0,\infty)$ and the second ones over all balls $B \subset \mathbb{R}^n$.

The function $\varphi(\cdot,t)$ is called to satisfy the *uniformly reverse Hölder condition for some* $q \in (1,\infty]$, denoted by $\varphi \in \mathbb{RH}_q(\mathbb{R}^n)$, if, when $q \in (1,\infty)$,

$$\mathbb{RH}_q(\varphi) : = \sup_{t\in[0,\infty)} \sup_{B\subset\mathbb{R}^n} \left\{ \frac{1}{|B|} \int_B [\varphi(x,t)]^q\,dx \right\}^{1/q} \left\{ \frac{1}{|B|} \int_B \varphi(x,t)\,dx \right\}^{-1} < \infty,$$

or

$$\mathbb{RH}_\infty(\varphi) := \sup_{t\in[0,\infty)} \sup_{B\subset\mathbb{R}^n} \left\{ \operatorname*{esssup}_{y\in B} \varphi(y,t) \right\} \left\{ \frac{1}{|B|} \int_B \varphi(x,t)\,dx \right\}^{-1} < \infty.$$

Here the first supremums are taken over all $t \in [0,\infty)$ and the second ones over all balls $B \subset \mathbb{R}^n$.

Recall that in Definition 2.1, $\mathbb{A}_q(\mathbb{R}^n)$ with $q \in [1,\infty)$ was introduced by Ky [34].

Let $\mathbb{A}_\infty(\mathbb{R}^n) := \cup_{q\in[1,\infty)} \mathbb{A}_q(\mathbb{R}^n)$ and define the *critical indices* of $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$ as follows:

$$q(\varphi) := \inf\{q \in [1,\infty) :\ \varphi \in \mathbb{A}_q(\mathbb{R}^n)\} \tag{2.3}$$

and

$$r(\varphi) := \sup\{q \in (1,\infty] :\ \varphi \in \mathbb{RH}_q(\mathbb{R}^n)\}. \tag{2.4}$$

Observe that if $q(\varphi) \in (1,\infty)$, then $\varphi \notin \mathbb{A}_{q(\varphi)}(\mathbb{R}^n)$, and there exists $\varphi \notin \mathbb{A}_1(\mathbb{R}^n)$ such that $q(\varphi) = 1$ (see, for example, [33]). Similarly, if $r(\varphi) \in (1,\infty)$, then $\varphi \notin \mathbb{RH}_{r(\varphi)}(\mathbb{R}^n)$, and there exists $\varphi \notin \mathbb{RH}_\infty(\mathbb{R}^n)$ such that $r(\varphi) = \infty$ (see, for example, [16]).

Now we introduce the notion of growth functions.

Definition 2.2. A function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called a *growth function* if the following hold:

(i) φ is a *Musielak-Orlicz function*, namely,

(i)$_1$ the function $\varphi(x,\cdot) : [0,\infty) \to [0,\infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$;

(i)$_2$ the function $\varphi(\cdot,t)$ is a measurable function for all $t \in [0,\infty)$.

(ii) $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$.

(iii) The function φ is of positive uniformly lower type p for some $p \in (0,1]$ and of uniformly upper type 1.

Clearly, $\varphi(x,t) := \omega(x)\Phi(t)$ is a growth function if $\omega \in A_\infty(\mathbb{R}^n)$ and Φ is an Orlicz function of lower type p for some $p \in (0,1]$ and of upper type 1. It is known that, for $p \in (0,1]$, if $\Phi(t) := t^p$ for all $t \in [0,\infty)$, then Φ is an Orlicz function of lower type p and of upper type p; for $p \in [\frac{1}{2},1]$, if $\Phi(t) := t^p/\ln(e+t)$ for all $t \in [0,\infty)$, then Φ is an Orlicz function of lower type q for $q \in (0,p)$ and of upper type p; for $p \in (0,\frac{1}{2}]$, if $\Phi(t) := t^p\ln(e+t)$ for all $t \in [0,\infty)$, then Φ is an Orlicz function of lower type p and of upper type q for $q \in (p,1]$. Recall that if an Orlicz function is of upper type $p \in (0,1)$, then it is also of upper type 1. Another typical and useful growth function is $\varphi(x,t) := \frac{t^\alpha}{[\ln(e+|x|)]^\beta+[\ln(e+t)]^\gamma}$ for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$ with any $\alpha \in (0,1]$, $\beta \in [0,\infty)$ and $\gamma \in [0,2\alpha(1+\ln 2)]$; more precisely, $\varphi \in \mathbb{A}_1(\mathbb{R}^n)$, φ is of uniformly upper type α and $i(\varphi) = \alpha$ which is not attainable (see [34]).

Throughout the whole paper, we *always assume that φ is a growth function* as in Definition 2.2. Let us now introduce the Musielak-Orlicz space.

The *Musielak-Orlicz space* $L^\varphi(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that $\int_{\mathbb{R}^n} \varphi(x,|f(x)|)\,dx < \infty$ with *Luxembourg norm*

$$\|f\|_{L^\varphi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

In what follows, for any measurable subset E of $\mathbb{R}^n$ and $t \in [0,\infty)$, we let

$$\varphi(E,t) := \int_E \varphi(x,t)\,dx.$$

The following Lemma 2.3 on the properties of growth functions is just [34, Lemmas 4.1 and 4.2].

Lemma 2.3. (i) *Let φ be a growth function. Then φ is uniformly σ-quasi-subadditive on $\mathbb{R}^n \times [0,\infty)$, namely, there exists a positive constant C such that for all $(x,t_j) \in \mathbb{R}^n \times [0,\infty)$ with $j \in \mathbb{N}$, $\varphi(x,\sum_{j=1}^\infty t_j) \le C\sum_{j=1}^\infty \varphi(x,t_j)$.*

(ii) *Let φ be a growth function and $\widetilde{\varphi}(x,t) := \int_0^t \frac{\varphi(x,s)}{s}\,ds$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$. Then $\widetilde{\varphi}$ is a growth function, which is equivalent to φ; moreover, $\widetilde{\varphi}(x,\cdot)$ is continuous and strictly increasing.*

(iii) *Let φ be a growth function. Then $\int_{\mathbb{R}^n} \varphi(x, \frac{|f(x)|}{\|f\|_{L^\varphi(\mathbb{R}^n)}}) = 1$ for all $f \in L^\varphi(\mathbb{R}^n) \setminus \{0\}$.*

We have the following properties for $\mathbb{A}_\infty(\mathbb{R}^n)$, whose proofs are similar to those in [23, 24].

Lemma 2.4. (i) *$\mathbb{A}_1(\mathbb{R}^n) \subset \mathbb{A}_p(\mathbb{R}^n) \subset \mathbb{A}_q(\mathbb{R}^n)$ for $1 \le p \le q < \infty$.*

(ii) *$\mathbb{RH}_\infty(\mathbb{R}^n) \subset \mathbb{RH}_p(\mathbb{R}^n) \subset \mathbb{RH}_q(\mathbb{R}^n)$ for $1 < q \le p \le \infty$.*

(iii) *If $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$ with $p \in (1,\infty)$, then there exists $q \in (1,p)$ such that $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$.*

(iv) *$\mathbb{A}_\infty(\mathbb{R}^n) = \cup_{p\in[1,\infty)}\mathbb{A}_p(\mathbb{R}^n) = \cup_{q\in(1,\infty]}\mathbb{RH}_q(\mathbb{R}^n)$.*

(v) *If $p \in (1, \infty)$ and $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$, then there exists a positive constant C such that for all measurable functions f on $\mathbb{R}^n$ and $t \in [0, \infty)$,*

$$\int_{\mathbb{R}^n} [\mathcal{M}(f)(x)]^p \varphi(x,t)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p \varphi(x,t)\,dx,$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal function on $\mathbb{R}^n$, defined by setting, for all $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)|\,dy,$$

where the supremum is taken over all balls $B \ni x$.

(vi) *If $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$ with $p \in [1, \infty)$, then there exists a positive constant C such that for all balls B_1, $B_2 \subset \mathbb{R}^n$ with $B_1 \subset B_2$ and $t \in [0, \infty)$,*

$$\frac{\varphi(B_2, t)}{\varphi(B_1, t)} \le C \left[\frac{|B_2|}{|B_1|}\right]^p.$$

(vii) *If $\varphi \in \mathbb{RH}_q(\mathbb{R}^n)$ with $q \in (1, \infty]$, then there exists a positive constant C such that for all balls B_1, $B_2 \subset \mathbb{R}^n$ with $B_1 \subset B_2$ and $t \in [0, \infty)$,*

$$\frac{\varphi(B_2, t)}{\varphi(B_1, t)} \ge C \left[\frac{|B_2|}{|B_1|}\right]^{(q-1)/q}.$$

3 Musielak-Orlicz tent spaces

In this section, we study the tent spaces associated with the growth function φ as in Definition 2.2. We first recall some notions as follows.

Let $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$. For any $\nu \in (0, \infty)$ and $x \in \mathbb{R}^n$, let

$$\Gamma_\nu(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ :\ |x - y| < \nu t\}$$

be the *cone of aperture ν with vertex $x \in \mathbb{R}^n$*. For any closed set F of $\mathbb{R}^n$, denote by $\mathcal{R}_\nu F$ the *union of all cones with vertices in F*, namely, $\mathcal{R}_\nu F := \cup_{x \in F} \Gamma_\nu(x)$ and, for any open set O in $\mathbb{R}^n$, the *tent over O* by $T_\nu(O)$, which is defined as $T_\nu(O) := [\mathcal{R}_\nu(O^\complement)]^\complement$. It is easy to see that

$$T_\nu(O) = \left\{(x,t) \in \mathbb{R}^{n+1}_+ :\ d(x, O^\complement) \ge \nu t\right\}.$$

In what follows, we denote $\Gamma_1(x)$ and $T_1(O)$ simply by $\Gamma(x)$ and $\widehat{O}$, respectively.

For all measurable functions g on $\mathbb{R}^{n+1}_+$ and $x \in \mathbb{R}^n$, define

$$\mathcal{A}(g)(x) := \left\{\int_{\Gamma(x)} |g(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}.$$

We remark that Coifman, Meyer and Stein [14] studied the tent space $T^p_2(\mathbb{R}^{n+1}_+)$ for $p \in (0, \infty)$. Recall that a measurable function g is said to belong to the *tent space* $T^p_2(\mathbb{R}^{n+1}_+)$

with $p \in (0,\infty)$, if $\|g\|_{T_2^p(\mathbb{R}_+^{n+1})} := \|\mathcal{A}(g)\|_{L^p(\mathbb{R}^n)} < \infty$. Moreover, the tent spaces $T_\Phi(\mathbb{R}_+^{n+1})$ associated with the Orlicz function Φ were studied in [25, 30].

Let φ be as in Definition 2.2. In what follows, we denote by $T_\varphi(\mathbb{R}_+^{n+1})$ the *space* of all measurable functions g on $\mathbb{R}_+^{n+1}$ such that $\mathcal{A}(g) \in L^\varphi(\mathbb{R}^n)$ and, for any $g \in T_\varphi(\mathbb{R}_+^{n+1})$, define its *quasi-norm* by

$$\|g\|_{T_\varphi(\mathbb{R}_+^{n+1})} := \|\mathcal{A}(g)\|_{L^\varphi(\mathbb{R}^n)} = \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \varphi\left(x, \frac{\mathcal{A}(g)(x)}{\lambda}\right) dx \le 1\right\}.$$

Let $p \in (1,\infty)$. A function a on $\mathbb{R}_+^{n+1}$ is called a $(\varphi,\, p)$-*atom* if

(i) there exists a ball $B \subset \mathbb{R}^n$ such that $\operatorname{supp} a \subset \widehat{B}$;

(ii) $\|a\|_{T_2^p(\mathbb{R}_+^{n+1})} \le |B|^{1/p} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}$.

Furthermore, if a is a (φ, p)-atom for all $p \in (1,\infty)$, we then call a a (φ,∞)-*atom*.

For functions in the space $T_\varphi(\mathbb{R}_+^{n+1})$, we have the following atomic decomposition.

Theorem 3.1. *Let φ be as in Definition 2.2. Then for any $f \in T_\varphi(\mathbb{R}_+^{n+1})$, there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{a_j\}_j$ of (φ,∞)-atoms such that for almost every $(x,t) \in \mathbb{R}_+^{n+1}$,*

$$f(x,t) = \sum_j \lambda_j a_j(x,t). \tag{3.1}$$

Moreover, there exists a positive constant C such that, for all $f \in T_\varphi(\mathbb{R}_+^{n+1})$,

$$\Lambda(\{\lambda_j a_j\}_j) := \inf\left\{\lambda \in (0,\infty) : \sum_j \varphi\left(B_j, \frac{|\lambda_j|}{\lambda \|\chi_{B_j}\|_{L^\varphi(\mathbb{R}^n)}}\right) \le 1\right\} \le C\|f\|_{T_\varphi(\mathbb{R}_+^{n+1})}, \tag{3.2}$$

where, for each j, $\widehat{B_j}$ appears in the support of a_j.

The proof of Theorem 3.1 is similar to that of [30, Theorem 3.1] (see also [14]). To this end, we need some known facts as follows.

Let F be a closed subset of $\mathbb{R}^n$ and $O := F^\complement$. Assume that $|O| < \infty$. For any fixed $\gamma \in (0,1)$, $x \in \mathbb{R}^n$ is said to have the *global γ-density* with respect to F if, for all $r \in (0,\infty)$,

$$\frac{|B(x,r) \cap F|}{|B(x,r)|} \ge \gamma.$$

Denote by F_γ^* the *set of all such x*. It is easy to prove that F_γ^* with $\gamma \in (0,1)$ is a closed subset of F. Let $\gamma \in (0,1)$ and $O_\gamma^* := (F_\gamma^*)^\complement$. Then O_γ^* is open and $O \subset O_\gamma^*$. Indeed, from the definition of O^*, we deduce that $O_\gamma^* = \{x \in \mathbb{R}^n : \mathcal{M}(\chi_O)(x) > 1-\gamma\}$, which, together with the fact that $\mathcal{M}$ is of type weak $(1,1)$, further implies that there exists a positive constant $C(\gamma)$, depending on γ, such that $|O_\gamma^*| \le C(\gamma)|O|$.

The following Lemma 3.2 is just [30, Lemma 3.1].

Lemma 3.2. *Let $\nu, \eta \in (0,\infty)$. Then there exist positive constants $\gamma \in (0,1)$ and $C(\gamma,\nu,\eta)$ such that for any closed subset F of $\mathbb{R}^n$ whose complement has finite measure, and any nonnegative measurable function H on $\mathbb{R}^{n+1}_+$,*

$$\int_{\mathcal{R}_\nu(F^*_\gamma)} H(y,t)t^n\,dy\,dt \le C(\gamma,\nu,\eta)\int_F \left\{\int_{\Gamma_\eta(x)} H(y,t)\,dy\,dt\right\}dx,$$

where F^*_γ denotes the set of points in $\mathbb{R}^n$ with the global γ-density with respect to F.

Moreover, we also need the following Lemma 3.3, whose proof is similar to that of [34, Lemma 5.4]. We omit the details.

Lemma 3.3. *Let φ be as in Definition 2.2, $f \in T_\varphi(\mathbb{R}^{n+1}_+)$, $k \in \mathbb{Z}$ and*

$$\Omega_k := \left\{x \in \mathbb{R}^n : \mathcal{A}(f)(x) > 2^k\right\}.$$

Then there exists a positive constant C such that, for all $\lambda \in (0,\infty)$,

$$\sum_{k\in\mathbb{Z}} \varphi\left(\Omega_k, \frac{2^k}{\lambda}\right) \le C\int_{\mathbb{R}^n} \varphi\left(x, \frac{\mathcal{A}(f)(x)}{\lambda}\right)dx.$$

Now we prove Theorem 3.1 by using Lemmas 3.2 and 3.3.

Proof of Theorem 3.1. Let $f \in T_\varphi(\mathbb{R}^{n+1}_+)$. For any $k \in \mathbb{Z}$, let

$$O_k := \left\{x \in \mathbb{R}^n : \mathcal{A}(f)(x) > 2^k\right\}$$

and $F_k := O_k^\complement$. Since $f \in T_\varphi(\mathbb{R}^{n+1}_+)$, for each k, O_k is an open set of $\mathbb{R}^n$ and $|O_k| < \infty$.

Let $\gamma \in (0,1)$ be as in Lemma 3.2 with $\eta = 1 = \nu$. In what follows, we denote $(F_k)^*_\gamma$ and $(O_k)^*_\gamma$ simply by F^*_k and O^*_k, respectively. We claim that $\operatorname{supp} f \subset (\cup_{k\in\mathbb{Z}}\widehat{O^*_k} \cup E)$, where $E \subset \mathbb{R}^{n+1}_+$ satisfies that $\int_E \frac{dy\,dt}{t} = 0$. Indeed, let $(x,t) \in \mathbb{R}^{n+1}_+$ be the Lebesgue point of f and $(x,t) \notin \cup_{k\in\mathbb{Z}}\widehat{O^*_k}$. Then there exists a sequence $\{y_k\}_{k\in\mathbb{Z}}$ of points such that $\{y_k\}_{k\in\mathbb{Z}} \subset B(x,t)$ and for each k, $y_k \notin \widehat{O^*_k}$, which implies that for each $k \in \mathbb{Z}$, $\mathcal{M}(\chi_{O_k})(y_k) \le 1-\gamma$. From this, we further deduce that

$$|B(x,t) \cap \{z \in \mathbb{R}^n : \mathcal{A}(f)(z) \le 2^k\}| \ge \gamma|B(x,t)|.$$

Let $k \to -\infty$. Then $|B(x,t) \cap \{z \in \mathbb{R}^n : \mathcal{A}(f)(z) = 0\}| \ge \gamma|B(x,t)|$. Therefore, there exists $y \in B(x,t)$ such that $f = 0$ almost everywhere in $\Gamma(y)$, which, together with Lebesgue's differentiation theorem, implies that $f(x,t) = 0$. From this, we infer that the claim holds.

Recall that O^*_k, for each $k \in \mathbb{Z}$, is open. Moreover, for each $k \in \mathbb{Z}$, by applying the Whitney decomposition to the set O^*_k, we obtain a set I_k of indices and a family $\{Q_{k,j}\}_{j\in I_k}$ of closed cubes with disjoint interiors such that

(i) $\cup_{j\in I_k} Q_{k,j} = O^*_k$ and, if $i \ne j$, then $Q_{k,j} \cap Q_{k,i} = \emptyset$;

(ii) $\sqrt{n}\ell(Q_{k,j}) \le \operatorname{dist}(Q_{k,j}, (O_k^*)^\complement) \le 4\sqrt{n}\ell(Q_{k,j})$, where $\ell(Q_{k,j})$ denotes the *side-length* of $Q_{k,j}$ and $\operatorname{dist}(Q_{k,j}, (O_k^*)^\complement) := \inf\{d(u,w) :\ u \in Q_{k,j},\ w \in (O_k^*)^\complement\}$.

Then for each $j \in I_k$, we let $B_{k,j}$ be the *ball with the center same as* $Q_{k,j}$ *and with the radius* $\frac{11}{2}\sqrt{n}$*-times* $\ell(Q_{k,j})$. Let $A_{k,j} := \widehat{B_{k,j}} \cap (Q_{k,j} \times (0,\infty)) \cap (O_k^* \setminus O_{k+1}^*)$,

$$a_{k,j} := 2^{-k}\|\chi_{B_{k,j}}\|^{-1}_{L^\varphi(\mathbb{R}^n)} f\chi_{k,j}$$

and $\lambda_{k,j} := 2^k\|\chi_{B_{k,j}}\|_{L^\varphi(\mathbb{R}^n)}$. Notice that $\{(Q_{k,j} \times (0,\infty)) \cap (O_k^* \setminus O_{k+1}^*)\} \subset \widehat{B_{k,j}}$. From this, we deduce that $f = \sum_{k\in\mathbb{Z}}\sum_{j\in I_k} \lambda_{k,j}a_{k,j}$ almost everywhere on $\mathbb{R}^{n+1}_+$.

We first show that for each $k \in \mathbb{Z}$ and $j \in I_k$, $a_{k,j}$ is a (φ,∞)-atom supported in $\widehat{B_{k,j}}$. Let $p \in (1,\infty)$, p' be its *conjugate index*, and $h \in T_2^{p'}(\mathbb{R}^{n+1}_+)$ with $\|h\|_{T_2^{p'}(\mathbb{R}^{n+1}_+)} \le 1$. Since $A_{k,j} \subset (\widehat{O_{k+1}^*})^\complement = F_{k+1}^*$, by Lemma 3.2 and Hölder's inequality, we see that

$$\begin{aligned}
|\langle a_{k,j}, h\rangle| &:= \left|\int_{\mathbb{R}^{n+1}_+} a_{k,j}(y,t)\chi_{A_{k,j}}(y,t)h(y,t)\frac{dy\,dt}{t}\right| \\
&\lesssim \int_{F_{k+1}}\int_{\Gamma(x)} |a_{k,j}(y,t)h(y,t)|\frac{dy\,dt}{t^{n+1}}\,dx \lesssim \int_{(O_{k+1})^\complement} \mathcal{A}(a_{k,j})(x)\mathcal{A}(h)(x)\,dx \\
&\lesssim 2^{-k}\|\chi_{B_{k,j}}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\left\{\int_{B_{k,j}\cap(O_{k+1})^\complement}[\mathcal{A}(f)(x)]^p\,dx\right\}^{1/p}\|h\|_{T_2^{p'}(\mathbb{R}^{n+1}_+)} \\
&\lesssim |B_{k,j}|^{1/p}\|\chi_{B_{k,j}}\|^{-1}_{L^\varphi(\mathbb{R}^n)},
\end{aligned}$$

which, together with $(T_2^p(\mathbb{R}^{n+1}_+))^* = T_2^{p'}(\mathbb{R}^{n+1}_+)$ (see [14]), where $(T_2^p(\mathbb{R}^{n+1}_+))^*$ denotes the *dual space* of $T_2^p(\mathbb{R}^{n+1}_+)$, implies that $\|a_{k,j}\|_{T_2^p(\mathbb{R}^{n+1}_+)} \lesssim |B_{k,j}|^{1/p}\|\chi_{B_{k,j}}\|^{-1}_{L^\varphi(\mathbb{R}^n)}$. Thus, $a_{k,j}$ is a (φ,p)-atom supported in $\widehat{B_{k,j}}$ up to a harmless constant for all $p \in (1,\infty)$ and hence a (φ,∞)-atom up to a harmless constant.

By Lemma 2.4(iv), we know that there exists $p_0 \in (q(\varphi),\infty)$ such that $\varphi \in \mathbb{A}_{p_0}(\mathbb{R}^n)$. From this and Lemma 2.4(v), it follows that, for any $k \in \mathbb{Z}$ and $t \in (0,\infty)$,

$$\begin{aligned}
\varphi(O_k^*, t) &\lesssim \frac{1}{(1-\gamma)^{p_0}}\int_{\mathbb{R}^n}[\mathcal{M}(\chi_{O_k})(x)]^{p_0}\,\varphi(x,t)\,dx \\
&\lesssim \frac{1}{(1-\gamma)^{p_0}}\int_{\mathbb{R}^n}[\chi_{O_k}(x)]^{p_0}\,\varphi(x,t)\,dx \sim \varphi(O_k,t),
\end{aligned}$$

which, together with Lemmas 2.3(i) and 3.3, implies that, for all $\lambda \in (0,\infty)$,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\sum_{j\in I_k}\varphi\left(B_{k,j}, \frac{|\lambda_{k,j}|}{\lambda\|\chi_{B_{k,j}}\|_{L^\varphi(\mathbb{R}^n)}}\right) &\lesssim \sum_{k\in\mathbb{Z}}\sum_{j\in I_k}\varphi\left(B_{k,j}, \frac{2^k}{\lambda}\right) \lesssim \sum_{k\in\mathbb{Z}}\varphi\left(O_k^*, \frac{2^k}{\lambda}\right) \\
&\lesssim \sum_{k\in\mathbb{Z}}\varphi\left(O_k, \frac{2^k}{\lambda}\right) \lesssim \int_{\mathbb{R}^n}\varphi\left(x, \frac{\mathcal{A}(f)(x)}{\lambda}\right)dx.
\end{aligned}$$

By this, we conclude that $\Lambda(\{\lambda_{k,j}a_{k,j}\}_{k\in\mathbb{Z},\,j}) \lesssim \|f\|_{T_\varphi(\mathbb{R}^{n+1}_+)}$, which completes the proof of Theorem 3.1. $\square$

Corollary 3.4. *Let $p \in (0,\infty)$ and φ be as in Definition 2.2. If $f \in T_\varphi(\mathbb{R}^{n+1}_+) \cap T^p_2(\mathbb{R}^{n+1}_+)$, then the decomposition* (3.1) *also holds in both $T_\varphi(\mathbb{R}^{n+1}_+)$ and $T^p_2(\mathbb{R}^{n+1}_+)$.*

Proof. Let $f \in T_\varphi(\mathbb{R}^{n+1}_+) \cap T^p_2(\mathbb{R}^{n+1}_+)$. We first show that (3.1) holds in $T_\varphi(\mathbb{R}^{n+1}_+)$. To this end, we need to prove that

$$\int_{\mathbb{R}^n} \varphi\left(x, \mathcal{A}(\lambda_{k,j} a_{k,j})(x)\right) dx \lesssim \varphi\left(B_{k,j}, \frac{|\lambda_{k,j}|}{\|\chi_{B_{k,j}}\|_{L^\varphi(\mathbb{R}^n)}}\right), \tag{3.3}$$

where for each k and j, $\lambda_{k,j}$, $a_{k,j}$ and $B_{k,j}$ are as in the proof of Theorem 3.1. Indeed, by $\operatorname{supp} a_{k,j} \subset \widehat{B_{k,j}}$, we know that $\operatorname{supp}(\mathcal{A}(\lambda_{k,j} a_{k,j})) \subset B_{k,j}$. Furthermore, by $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$ and Lemma 2.4(iv), we see that there exists $q_0 \in (1,\infty)$ such that $\varphi \in \mathbb{RH}_{q_0}(\mathbb{R}^n)$. From this, the uniformly upper type 1 property of φ, Hölder's inequality and that $a_{k,j}$ is a (φ,∞)-atom up to a harmless constant, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varphi\left(x, \mathcal{A}(\lambda_{k,j} a_{k,j})(x)\right) dx \\
&\quad\lesssim \int_{B_{k,j}} \left[1 + \mathcal{A}(a_{k,j})(x)\|\chi_{B_{k,j}}\|_{L^\varphi(\mathbb{R}^n)}\right] \varphi\left(x, \frac{|\lambda_{k,j}|}{\|\chi_{B_{k,j}}\|_{L^\varphi(\mathbb{R}^n)}}\right) dx \\
&\quad\lesssim \varphi\left(B_{k,j}, \frac{|\lambda_{k,j}|}{\|\chi_{B_{k,j}}\|_{L^\varphi(\mathbb{R}^n)}}\right) + \left\{\int_{B_{k,j}} [\mathcal{A}(a_{k,j})(x)]^{q_0'}\, dx\right\}^{1/q_0'} \|\chi_{B_{k,j}}\|_{L^\varphi(\mathbb{R}^n)} \\
&\qquad\times \left\{\int_{B_{k,j}} \left[\varphi\left(x, |\lambda_{k,j}|\|\chi_{B_{k,j}}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{q_0} dx\right\}^{1/q_0} \\
&\quad\lesssim \varphi\left(B_{k,j}, |\lambda_{k,j}|\|\chi_{B_{k,j}}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right) + \|a_{k,j}\|_{T^{q_0'}_2(\mathbb{R}^{n+1}_+)} \|\chi_{B_{k,j}}\|_{L^\varphi(\mathbb{R}^n)} \\
&\qquad\times |B_{k,j}|^{-1/q_0'} \varphi\left(B_{k,j}, |\lambda_{k,j}|\|\chi_{B_{k,j}}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right) \lesssim \varphi\left(B_{k,j}, |\lambda_{k,j}|\|\chi_{B_{k,j}}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right),
\end{aligned}$$

which implies that (3.3) holds. It was proved in Theorem 3.1 that

$$\sum_{k\in\mathbb{Z}} \sum_{j\in I_k} \varphi\left(B_{k,j}, \frac{|\lambda_{k,j}|}{\|\chi_{B_{k,j}}\|_{L^\varphi(\mathbb{R}^n)}}\right) \lesssim \int_{\mathbb{R}^n} \varphi(x, \mathcal{A}(f)(x))\, dx < \infty.$$

By this, (3.1) and Lemma 2.3(i), we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varphi\left(x, \mathcal{A}\left(f - \sum_{|k|+j<N} \lambda_{k,j} a_{k,j}\right)(x)\right) dx \\
&\quad\lesssim \sum_{|k|+j\ge N} \int_{\mathbb{R}^n} \varphi\left(x, \mathcal{A}(\lambda_{k,j} a_{k,j})(x)\right) dx \lesssim \sum_{|k|+j\ge N} \varphi\left(B_{k,j}, \frac{|\lambda_{k,j}|}{\|\chi_{B_{k,j}}\|_{L^\varphi(\mathbb{R}^n)}}\right) \to 0,
\end{aligned}$$

as $N \to \infty$. Therefore, (3.1) holds in $T_\varphi(\mathbb{R}^{n+1}_+)$. Moreover, similar to the proof of [30, Proposition 3.1], we know that (3.1) also holds in $T^p_2(\mathbb{R}^{n+1}_+)$, which completes the proof of Corollary 3.4. ∎

In what follows, let $T^c_\varphi(\mathbb{R}^{n+1}_+)$ and $T^{p,\,c}_2(\mathbb{R}^{n+1}_+)$ with $p\in(0,\infty)$ denote, respectively, the *sets of all functions in* $T_\varphi(\mathbb{R}^{n+1}_+)$ *and* $T^p_2(\mathbb{R}^{n+1}_+)$ *with compact support.*

Proposition 3.5. *Let φ be as in Definition 2.2. Then $T^c_\varphi(\mathbb{R}^{n+1}_+)\subset T^{2,\,c}_2(\mathbb{R}^{n+1}_+)$ as sets.*

Proof. It is well known that for all $p\in(0,\infty)$, $T^{p,\,c}_2(\mathbb{R}^{n+1}_+)\subset T^{2,\,c}_2(\mathbb{R}^{n+1}_+)$ as sets (see, for example, [30, Lemma 3.3(i)]). Thus, to prove $T^c_\varphi(\mathbb{R}^{n+1}_+)\subset T^{2,\,c}_2(\mathbb{R}^{n+1}_+)$, it suffices to show that $T^c_\varphi(\mathbb{R}^{n+1}_+)\subset T^{p,\,c}_2(\mathbb{R}^{n+1}_+)$ for some $p\in(0,\infty)$. Suppose that $f\in T^c_\varphi(\mathbb{R}^{n+1}_+)$ and $\operatorname{supp} f\subset K$, where K is a compact set in $\mathbb{R}^{n+1}_+$. Let B be a ball in $\mathbb{R}^n$ such that $K\subset\widehat{B}$. Then $\operatorname{supp}(\mathcal{A}(f))\subset\widehat{B}$. Let $p_0\in(0,i(\varphi))$ and $q_0\in(q(\varphi),\infty)$. Then φ is of uniformly lower type p_0 and $\varphi\in\mathbb{A}_{q_0}(\mathbb{R}^n)$. From this, Hölder's inequality, (2.2) and the uniformly lower type p_0 property of φ, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^n}[\mathcal{A}(f)(x)]^{p_0/q_0}\,dx\\
&\quad\le\left\{\int_B[\mathcal{A}(f)(x)]^{p_0}\varphi(x,1)\,dx\right\}^{1/q_0}\left\{\int_B[\varphi(x,1)]^{-q_0'/q_0}\,dx\right\}^{1/q_0'}\\
&\quad\lesssim\frac{|B|}{[\varphi(B,1)]^{1/q_0}}\left\{\int_{\{x\in B:\ \mathcal{A}(f)(x)\le1\}}[\mathcal{A}(f)(x)]^{p_0}\varphi(x,1)\,dx+\int_{\{x\in B:\ \mathcal{A}(f)(x)>1\}}\cdots\right\}^{1/q_0}\\
&\quad\lesssim\frac{|B|}{[\varphi(B,1)]^{1/q_0}}\left\{\varphi(B,1)+\int_B\varphi(x,\mathcal{A}(f)(x))\,dx\right\}^{1/q_0}<\infty,
\end{aligned}$$

where $1/q_0+1/q_0'=1$, which, implies that $f\in T^{p_0/q_0,\,c}_2(\mathbb{R}^{n+1}_+)\subset T^{2,\,c}_2(\mathbb{R}^{n+1}_+)$. This finishes the proof of Proposition 3.5. $\square$

4 Lusin area function and molecular characterizations of $H_\varphi(\mathbb{R}^n)$

In this section, we first recall the Musielak-Orlicz Hardy space $H_\varphi(\mathbb{R}^n)$ introduced by Ky [34]. Then we establish two equivalent characterizations of $H_\varphi(\mathbb{R}^n)$ in terms of the molecule and the Lusin area function. We begin with some notions and notation.

In what follows, we denote by $\mathcal{S}(\mathbb{R}^n)$ the *space of all Schwartz functions* and by $\mathcal{S}'(\mathbb{R}^n)$ its *dual space* (namely, the *space of all tempered distributions*). For $m\in\mathbb{N}$, define

$$\mathcal{S}_m(\mathbb{R}^n):=\left\{\phi\in\mathcal{S}(\mathbb{R}^n):\ \sup_{x\in\mathbb{R}^n}\ \sup_{\beta\in\mathbb{Z}^n_+,\,|\beta|\le m+1}(1+|x|)^{(m+2)(n+1)}|\partial^\beta_x\phi(x)|\le1\right\}.$$

Then for all $f\in\mathcal{S}'(\mathbb{R}^n)$, the *non-tangential grand maximal function* f^*_m of f is defined by setting, for all $x\in\mathbb{R}^n$,

$$f^*_m(x):=\sup_{\phi\in\mathcal{S}_m(\mathbb{R}^n)}\ \sup_{|y-x|<t,\,t\in(0,\infty)}|f*\phi_t(y)|,$$

where for all $t \in (0,\infty)$, $\phi_t(\cdot) := t^{-n}\phi(\frac{\cdot}{t})$. When $m(\varphi) := \lfloor n[q(\varphi)/i(\varphi) - 1]\rfloor$, where $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.3) and (2.1), we denote $f^*_{m(\varphi)}$ simply by f^*.

Now we recall the definition of the Musielak-Orlicz Hardy $H_\varphi(\mathbb{R}^n)$ introduced by Ky [34] as follows.

Definition 4.1. Let φ be as in Definition 2.2. The *Musielak-Orlicz Hardy space* $H_\varphi(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f^* \in L^\varphi(\mathbb{R}^n)$ with the *quasi-norm* $\|f\|_{H_\varphi(\mathbb{R}^n)} := \|f^*\|_{L^\varphi(\mathbb{R}^n)}$.

Definition 4.2. Let φ be as in Definition 2.2 and $\alpha \in (0,\infty)$. Assume that $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a radial real-valued function satisfying that

$$\int_{\mathbb{R}^n} \phi(x)x^\gamma \, dx = 0 \tag{4.1}$$

for all $\gamma \in \mathbb{Z}^n_+$ with $|\gamma| \le s$, where $s \in \mathbb{Z}_+$ with $s \ge \lfloor n[q(\varphi)/i(\varphi) - 1]\rfloor$, and

$$\int_0^\infty |\widehat{\phi}(t\xi)|^2 \frac{dt}{t} = 1$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$, where $\widehat{\phi}$ denotes the *Fourier transform* of ϕ.

Then for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define

$$S_\alpha(f)(x) := \left\{\int_{\Gamma_\alpha(x)} |\phi_t * f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}.$$

Moreover, when $\alpha = 1$, denote $S_1(f)$ simply by $S(f)$.

It is known that the Lusin area function S is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$ (see, for example, [20]).

Now we introduce the Musielak-Orlicz Hardy space $H_{\varphi,S}(\mathbb{R}^n)$ via the Lusin area function as follows.

Definition 4.3. Let φ be as in Definition 2.2. The *Musielak-Orlicz Hardy space* $H_{\varphi,S}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $S(f) \in L^\varphi(\mathbb{R}^n)$ with the *quasi-norm*

$$\|f\|_{H_{\varphi,S}(\mathbb{R}^n)} := \|S(f)\|_{L^\varphi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \varphi\left(x, \frac{S(f)(x)}{\lambda}\right) dx \le 1\right\}.$$

To introduce the molecular Musielak-Orlicz Hardy space, we first introduce the notion of the molecule associated with the growth function φ.

Definition 4.4. Let φ be as in Definition 2.2, $q \in (1,\infty)$, $s \in \mathbb{Z}_+$ and $\varepsilon \in (0,\infty)$. A function $\alpha \in L^q(\mathbb{R}^n)$ is called a $(\varphi, q, s, \varepsilon)$*-molecule* associated with the ball B if

(i) for each $j \in \mathbb{Z}_+$, $\|\alpha\|_{L^q(U_j(B))} \le 2^{-j\varepsilon}|2^jB|^{1/q}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}$, where $U_0(B) := B$ and $U_j(B) := 2^jB \setminus 2^{j-1}B$ with $j \in \mathbb{N}$;

(ii) $\int_{\mathbb{R}^n} \alpha(x)x^\beta \, dx = 0$ for all $\beta \in \mathbb{Z}^n_+$ with $|\beta| \le s$.

Definition 4.5. Let φ be as in Definition 2.2, $p,\,q\in(1,\infty)$, $s\in\mathbb{Z}_+$ and $\varepsilon\in(0,\infty)$. The *molecular Musielak-Orlicz Hardy space*, $H^{q,s,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$, is defined to be the space of all $f\in\mathcal{S}'(\mathbb{R}^n)$ satisfying that $f=\sum_j\lambda_j\alpha_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{\lambda_j\}\subset\mathbb{C}$ and $\{\alpha_j\}_j$ is a sequence of $(\varphi,\,q,\,s,\,\varepsilon)$-molecules with

$$\sum_j\varphi\left(B_j,\|\chi_{B_j}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)<\infty,$$

where, for each j, the molecule α_j is associated with the ball B_j. Moreover, define

$$\|f\|_{H^{q,s,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)}:=\inf\left\{\Lambda\left(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}\right)\right\},$$

where the infimum is taken over all decompositions of f as above, and

$$\Lambda\left(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}\right):=\inf\left\{\lambda\in(0,\infty):\ \sum_{j\in\mathbb{N}}\varphi\left(B_j,\frac{|\lambda_j|}{\lambda\|\chi_{B_j}\|_{L^\varphi(\mathbb{R}^n)}}\right)\le 1\right\}.$$

Here, for each $j\in\mathbb{N}$, α_j is associated with the ball B_j.

Definition 4.6. Let ϕ be as in Definition 4.2. For all $f\in T^{p,\,c}_2(\mathbb{R}^{n+1}_+)$ with $p\in(1,\infty)$ and $x\in\mathbb{R}^n$, define

$$\pi_\phi(f)(x):=\int_0^\infty\left(f(\cdot,t)*\phi_t\right)(x)\,\frac{dt}{t}. \tag{4.2}$$

It was proved in [14] that $\pi_\phi(f)\in L^2(\mathbb{R}^n)$ for such f. Moreover, we have the following properties for the operator π_ϕ.

Proposition 4.7. *Let ϵ and s be as in* (4.1)*, π_ϕ as in* (4.2) *and φ as in Definition 2.2. Then*

(i) *the operator π_ϕ, initially defined on the space $T^{p,\,c}_2(\mathbb{R}^{n+1}_+)$ with $p\in(1,\infty)$, extends to a bounded linear operator from $T^p_2(\mathbb{R}^{n+1}_+)$ to $L^p(\mathbb{R}^n)$;*

(ii) *the operator π_ϕ, initially defined on the space $T^c_\varphi(\mathbb{R}^{n+1}_+)$, extends to a bounded linear operator from $T_\varphi(\mathbb{R}^{n+1}_+)$ to $H_{\varphi,S}(\mathbb{R}^n)$.*

Proof. The conclusion (i) is just [14, Theorem 6(1)]. Now we prove (ii). Let $f\in T^c_\varphi(\mathbb{R}^{n+1}_+)$. Then by Proposition 3.5, Corollary 3.4 and (i), we know that

$$\pi_\phi(f)=\sum_j\lambda_j\pi_\phi(a_j)=:\sum_j\lambda_j\alpha_j$$

in $L^2(\mathbb{R}^n)$, where $\{\lambda_j\}_j$ and $\{a_j\}_j$ satisfy (3.1) and (3.2). Recall that for each j, $\operatorname{supp}a_j\subset\widehat{B_j}$ and B_j is a ball of $\mathbb{R}^n$. Moreover, from the fact that S is bounded on $L^p(\mathbb{R}^n)$ with $p\in(1,\infty)$, we deduce that for all $x\in\mathbb{R}^n$, $S(\pi_\phi(f))(x)\le\sum_j|\lambda_j|S(\alpha_j)(x)$. This, combined with Lemma 2.3(i), yields that

$$\int_{\mathbb{R}^n}\varphi(x,S(\pi_\phi(f))(x))\,dx\lesssim\sum_j\int_{\mathbb{R}^n}\varphi(x,|\lambda_j|S(\alpha_j)(x))\,dx. \tag{4.3}$$

We now claim that for some $\varepsilon \in (0,\infty)$, $\alpha_j = \pi_\phi(a_j)$ is a $(\varphi,\infty,s,\varepsilon)$-molecule, up to a harmless constant, associated with the ball B_j for each j. Indeed, assume that a is a (φ,∞)-atom supported in the ball $B := B(x_B, r_B)$ and $q \in (1,\infty)$. Since for $q \in (1,2)$, each $(\varphi,2,s,\varepsilon)$-molecule is also a $(\varphi,q,s,\varepsilon)$-molecule, to prove the above claim, it suffices to show that $\alpha := \pi_\phi(a)$ is a $(\varphi,q,s,\varepsilon)$-molecule, up to a harmless constant, associated with B with $q \in [2,\infty)$.

Let $q \in [2,\infty)$. When $j \in \{0, \cdots, 4\}$, by (i), we know that

$$\|\alpha\|_{L^q(U_j(B))} = \|\pi_\phi(a)\|_{L^q(U_j(B))} \lesssim \|a\|_{T_2^q(\mathbb{R}^{n+1}_+)} \lesssim |B|^{1/q}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}. \tag{4.4}$$

When $j \in \mathbb{N}$ with $j \geq 4$, take $h \in L^{q'}(\mathbb{R}^n)$ satisfying $\|h\|_{L^{q'}(\mathbb{R}^n)} \leq 1$ and $\operatorname{supp}(h) \subset U_j(B)$. Then from Hölder's inequality and $q' \in (1,2]$, we infer that

$$\begin{aligned}|\langle \pi_\phi(a), h\rangle| &= \left|\int_{\mathbb{R}^n}\int_0^\infty (a(\cdot,t)*\phi_t)(x)\frac{dt}{t}h(x)\,dx\right| \\ &\leq \int_B\int_0^{r_B} |a(x,t)||\phi_t * h(x)|\,dx\,\frac{dt}{t} \\ &\lesssim \|\mathcal{A}(a)\|_{L^q(\mathbb{R}^n)}\left\|\mathcal{A}\left(\chi_{\widehat{B}}\phi_t * h\right)\right\|_{L^{q'}(\mathbb{R}^n)} \\ &\lesssim \|a\|_{T_2^q(\mathbb{R}^{n+1}_+)}|B|^{1/q'-1/2}\left\{\int_{\widehat{B}}|\phi_t * h(x)|^2\,\frac{dx\,dt}{t}\right\}^{1/2}.\end{aligned} \tag{4.5}$$

Let $\epsilon \in (n[q(\varphi)/i(\varphi)-1],\infty)$. Then by $\phi \in \mathcal{S}(\mathbb{R}^n)$, Hölder's inequality and the fact that for any $x \in B$ and $y \in U_j(B)$, $|x-y| \gtrsim 2^{j-1}r_B$, we conclude that, for all $x \in B$,

$$\begin{aligned}|\phi_t * h(x)| &\lesssim \int_{\mathbb{R}^n}\frac{t^\epsilon}{(t+|x-y|)^{n+\epsilon}}|h(y)|\,dy \\ &\lesssim \frac{t^\epsilon}{(2^jr_B)^{n+\epsilon}}\|h\|_{L^{q'}(\mathbb{R}^n)}|2^jB|^{1/q} \lesssim \frac{t^\epsilon}{(2^jr_B)^{n/q'+\epsilon}},\end{aligned}$$

which, together with (4.5), implies that

$$|\langle \pi_\phi(a), h\rangle| \lesssim 2^{-j(n+\epsilon)}|2^jB|^{1/q}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}.$$

From this and the choice of h, we deduce that, for each $j \in \mathbb{N}$ with $j \geq 4$,

$$\|\alpha_j\|_{L^q(U_j(B))} = \|\pi_\phi(a)\|_{L^q(U_j(B))} \lesssim 2^{-j(n+\epsilon)}|2^jB|^{1/q}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}. \tag{4.6}$$

Moreover, by (4.1), we know that, for all $\gamma \in \mathbb{Z}^n_+$ with $|\gamma| \leq s$,

$$\int_{\mathbb{R}^n}\pi_\phi(a)(x)x^\gamma\,dx = \int_0^\infty\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\phi_t(x-y)x^\gamma\,dx\right\}a(y,t)\,\frac{dy\,dt}{t} = 0,$$

which, together with (4.4) and (4.6), implies that α is a $(\varphi,q,s,n+\epsilon)$-molecule, up to a harmless constant, associated with B. Thus, the claim holds.

By $\epsilon > n[q(\varphi)/i(\varphi)-1]$ and $s \geq \lfloor n[q(\varphi)/i(\varphi)-1]\rfloor$, we know that there exist $p_0 \in (0,i(\varphi))$ and $q_0 \in (q(\varphi),\infty)$ such that $\epsilon > n(q_0/p_0-1)$ and $s+1 > n(q_0/p_0-1)$. Then

$\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$ and φ is of uniformly lower type p_0. Let $\varepsilon := n+\epsilon$ and $q \in [2,\infty)\cap(1/p_0,\infty)$ satisfying $q' < r(\varphi)$. Then $\varphi \in \mathbb{RH}_{q'}(\mathbb{R}^n)$. We now claim that, for any $\lambda \in \mathbb{C}$ and $(\varphi, q, s, \varepsilon)$-molecule α associated with the ball $B \subset \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \varphi(x, S(\lambda\alpha)(x))\,dx \lesssim \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right). \tag{4.7}$$

If (4.7) holds, from (4.7), the facts that for all $\lambda \in (0,\infty)$, $S(\pi_\phi(f/\lambda)) = S(\pi_\phi(f))/\lambda$ and $\pi_\phi(f/\lambda) = \sum_j \lambda_j\alpha_j/\lambda$, and $S(\pi_\phi(f)) \le \sum_j |\lambda_j| S(\alpha_j)$, it follows that, for all $\lambda \in (0,\infty)$,

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{S(\pi_\phi(f))(x)}{\lambda}\right)\,dx \lesssim \sum_j \varphi\left(B_j, \frac{|\lambda_j|}{\lambda\|\chi_{B_j}\|_{L^\varphi(\mathbb{R}^n)}}\right),$$

which, together with (3.2), implies that $\|\pi_\phi(f)\|_{H_{\varphi,S}(\mathbb{R}^n)} \lesssim \Lambda(\{\lambda_j\alpha_j\}_j) \lesssim \|f\|_{T_\varphi(\mathbb{R}^{n+1}_+)}$, and hence completes the proof of (ii).

Now we prove (4.7). For any $x \in \mathbb{R}^n$, by Hölder's inequality, the moment condition of ϕ and the Taylor remainder theorem, we see that

$$\begin{aligned}
S(\alpha)(x) &\le \left\{\int_0^{r_B}\int_{B(x,t)} |\phi_t * \alpha(y)|^2\,\frac{dy\,dt}{t^{n+1}}\right\}^{1/2} + \left\{\int_{r_B}^\infty \int_{B(x,t)} \cdots\right\}^{1/2} \\
&\le \sum_{j=0}^\infty \left\{\int_0^{r_B}\int_{B(x,t)} \left|\phi_t * \left(\alpha\chi_{U_j(B)}\right)(y)\right|^2\,\frac{dy\,dt}{t^{n+1}}\right\}^{1/2} \\
&\quad + \sum_{j=0}^\infty \sum_{\gamma\in\mathbb{Z}^n_+,\,|\gamma|=s+1} \left\{\int_{r_B}^\infty \int_{B(x,t)} \left[\int_{\mathbb{R}^n} \frac{1}{t^n}\right.\right. \\
&\quad \times \left|(\partial_x^\gamma\phi)\left(\frac{\theta(y-z)+(1-\theta)(y-x_B)}{t}\right)\right| \left|\frac{z-x_B}{t}\right|^{s+1} \\
&\quad \times \left.\left.\left|\left(\alpha\chi_{U_j(B)}\right)(z)\right|\,dz\right]^2\frac{dy\,dt}{t^{n+1}}\right\}^{1/2} =: \sum_{j=0}^\infty [\mathrm{E}_j(x)+\mathrm{F}_j(x)],
\end{aligned} \tag{4.8}$$

where $\theta \in (0,1)$. For any $j \in \mathbb{Z}_+$, let $B_j := 2^jB$. Then from (4.8) and Lemma 2.3(i), we infer that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varphi(x, S(\lambda\alpha)(x))\,dx \\
&\quad\lesssim \int_{\mathbb{R}^n} \varphi\left(x, |\lambda|\sum_{j=0}^\infty [\mathrm{E}_j(x)+\mathrm{F}_j(x)]\right)\,dx \\
&\quad\lesssim \sum_{j=0}^\infty \left\{\int_{\mathbb{R}^n} \varphi\left(x, |\lambda|\mathrm{E}_j(x)\right)\,dx + \int_{\mathbb{R}^n} \varphi\left(x, |\lambda|\mathrm{F}_j(x)\right)\,dx\right\} \\
&\quad\lesssim \sum_{j=0}^\infty\sum_{i=0}^\infty \left\{\int_{U_i(B_j)} \varphi(x, |\lambda|\mathrm{E}_j(x))\,dx + \int_{U_i(B_j)} \varphi(x, |\lambda|\mathrm{F}_j(x))\,dx\right\}
\end{aligned} \tag{4.9}$$

$$=: \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(\mathrm{E}_{i,j}+\mathrm{F}_{i,j}).$$

When $i \in \{0,\, 1,\, \cdots,\, 4\}$, by the uniformly upper type 1 and lower type p_0 properties of φ, we see that

$$\begin{aligned}\text{(4.10)}\qquad \mathrm{E}_{i,j} &\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)} \int_{U_i(B_j)} \varphi\left(x, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right) S\left(\chi_{U_j(B)}\alpha\right)(x)\,dx\\ &\quad + \|\chi_B\|^{p_0}_{L^\varphi(\mathbb{R}^n)} \int_{U_i(B_j)} \varphi\left(x, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right) \left[S\left(\chi_{U_j(B)}\alpha\right)(x)\right]^{p_0}\,dx\\ &=: \mathrm{G}_{i,j}+\mathrm{H}_{i,j}.\end{aligned}$$

Now we estimate $\mathrm{G}_{i,j}$. From Hölder's inequality, the $L^q(\mathbb{R}^n)$-boundedness of S, $\varphi \in \mathbb{RH}_{q'}(\mathbb{R}^n)$ and Lemma 2.4(vi), we deduce that

$$\begin{aligned}\text{(4.11)}\qquad \mathrm{G}_{i,j} &\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)} \left\{\int_{U_i(B_j)} \left[S\left(\chi_{U_j(B)}\alpha\right)(x)\right]^q\,dx\right\}^{1/q}\\ &\quad\times\left\{\int_{U_i(B_j)} \left[\varphi\left(x, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{q'}\,dx\right\}^{1/q'}\\ &\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)} \|\alpha\|_{L^q(U_j(B))} |2^{i+j}B|^{-1/q} \varphi\left(2^{i+j}B, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\\ &\lesssim 2^{-j[(n+\epsilon)-nq_0]} \varphi\left(B, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right).\end{aligned}$$

For $\mathrm{H}_{i,j}$, similarly, by $p_0 q \in (1,\infty)$, we have

$$\begin{aligned}\mathrm{H}_{i,j} &\lesssim \|\chi_B\|^{p_0}_{L^\varphi(\mathbb{R}^n)} \left\{\int_{U_i(B_j)} \left[S\left(\chi_{U_j(B)}\alpha\right)(x)\right]^{p_0 q}\,dx\right\}^{1/q}\\ &\quad\times\left\{\int_{U_i(B_j)} \left[\varphi\left(x, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{q'}\,dx\right\}^{1/q'}\\ &\lesssim \|\chi_B\|^{p_0}_{L^\varphi(\mathbb{R}^n)} \|\alpha\|^{p_0}_{L^{p_0 q}(U_j(B))} |2^{i+j}B|^{-1/q} \varphi\left(2^{i+j}B, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\\ &\lesssim 2^{-j[(n+\epsilon)p_0-nq_0]} \varphi\left(B, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right),\end{aligned}$$

which, together with (4.10) and (4.11), implies that, for each $j \in \mathbb{Z}_+$ and $i \in \{0,\, 1,\, \cdots,\, 4\}$,

$$\text{(4.12)}\qquad \mathrm{E}_{i,j} \lesssim 2^{-j[(n+\epsilon)p_0-nq_0]} \varphi\left(B, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right).$$

When $i \in \mathbb{N}$ with $i \geq 4$, by the uniformly upper type 1 and lower type p_0 properties of φ, we conclude that

$$\text{(4.13)}\quad \mathrm{E}_{i,j} \lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)} \int_{U_i(B_j)} \varphi\left(x, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right) \mathrm{E}_j(x)\,dx$$

$$+\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_0}\int_{U_i(B_j)}\varphi\left(x,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)[\mathrm{E}_j(x)]^{p_0}\,dx=:\mathrm{K}_{i,j}+\mathrm{J}_{i,j}.$$

For any given $x\in U_i(B_j)$ and $y\in B(x,t)$ with $t\in(0,r_B]$, we see that, for any $z\in U_j(B)$, $|y-z|\gtrsim 2^{i+j}r_B$. Then from $\phi\in\mathcal{S}(\mathbb{R}^n)$ and Hölder's inequality, it follows that

$$\begin{aligned}\left|\phi_t*\left(\alpha\chi_{U_j(B)}\right)(y)\right|&\lesssim\int_{U_j(B)}\frac{t^\epsilon}{(1+|y-z|)^{n+\epsilon}}|\alpha(z)|\,dz\\&\lesssim\frac{t^\epsilon}{(2^{i+j}r_B)^{n+\epsilon}}\|\alpha\|_{L^q(U_j(B))}|U_j(B)|^{1/q'},\end{aligned}$$

where ϵ is as in (4.6), which implies that, for all $x\in U_i(B_j)$,

$$\mathrm{E}_j(x)\lesssim\frac{r_B^\epsilon\|\alpha\|_{L^q(U_j(B))}|U_j(B)|^{1/q'}}{(2^{i+j}r_B)^{n+\epsilon}}\lesssim 2^{-i(n+\epsilon)}2^{-j(\epsilon+\varepsilon)}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}. \tag{4.14}$$

By this, Hölder's inequality and Lemma 2.4(vi), we see that

$$\begin{aligned}\mathrm{K}_{i,j}&\lesssim 2^{-i(n+\epsilon)}2^{-j(\epsilon+\varepsilon)}\varphi\left(2^{i+j}B,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)\\&\lesssim 2^{-i(n+\epsilon-nq_0)}2^{-j(\epsilon+\varepsilon-nq_0)}\varphi\left(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right).\end{aligned} \tag{4.15}$$

Now we estimate $\mathrm{J}_{i,j}$. From (4.14) and Lemma 2.4(vi), it follows that

$$\mathrm{J}_{i,j}\lesssim 2^{-ip_0(n+\epsilon-nq_0/p_0)}2^{-jp_0(\epsilon+\varepsilon-nq_0/p_0)}\varphi\left(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right). \tag{4.16}$$

By (4.13), (4.15) and (4.16), we know that, when $i\in\mathbb{N}$ with $i\ge 4$ and $j\in\mathbb{Z}_+$,

$$\mathrm{E}_{i,j}\lesssim 2^{-ip_0(n+\epsilon-nq_0/p_0)}2^{-jp_0(\epsilon+\varepsilon-nq_0/p_0)}\varphi\left(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right). \tag{4.17}$$

Now we deal with $\mathrm{F}_{i,j}$. When $i\in\{0,1,\cdots,4\}$, similar to the proof of (4.12), we see that

$$\mathrm{F}_{i,j}\lesssim 2^{-j[(n+\epsilon)p_0-nq_0]}\varphi\left(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right). \tag{4.18}$$

When $i\in\mathbb{N}$ with $i\ge 4$ and $j\in\mathbb{Z}_+$, for any $x\in U_i(B_j)$, $y\in B(x,t)$ with $t\in[r_B,2^{i+j-2}r_B)$ and $z\in U_j(B)$, we know that $|z-x_B|\le 2^jr_B$ and $|y-z|\ge|x-z|-|x-y|\ge 2^{i+j-1}r_B-t\ge 2^{i+j-3}r_B$. From these, we deduce that

$$|\theta(y-z)+(1-\theta)(y-x_B)|=|(y-z)-(1-\theta)(z-x_B)|\ge 2^{i+j-3}r_B-2^jr_B\ge 2^{i+j-4}r_B.$$

Thus, by this and (4.1), we know that, for all $\gamma\in\mathbb{Z}_+^n$ with $|\gamma|=s+1$,

$$\int_{r_B}^{2^{i+j-2}r_B}\int_{B(x,t)}\left\{\int_{\mathbb{R}^n}\frac{1}{t^n}\left|(\partial_x^\gamma\phi)\left(\frac{\theta(y-z)+(1-\theta)(y-x_B)}{t}\right)\right|\right. \tag{4.19}$$

$$\times \left|\frac{z-x_B}{t}\right|^{s+1} \left|\left(\alpha\chi_{U_j(B)}\right)(z)\right| \,dz\Bigg\}^2 \frac{dy\,dt}{t^{n+1}}$$

$$\lesssim \int_{r_B}^{2^{i+j-2}r_B} \int_{B(x,t)} \left\{\int_{U_j(B)} \frac{t^{n+s+1+\epsilon}}{(2^{i+j-4}r_B)^{n+1+s+\epsilon}} |z-x_B|^{s+1} |(\alpha\chi_{U_j(B)})(z)| \,dz\right\}^2$$

$$\times \frac{dy\,dt}{t^{2(n+s+1)+n+1}}$$

$$\lesssim (2^{i+j}r_B)^{-2(n+s+1+\epsilon)}(2^j r_B)^{2(s+1)} \|\alpha\|^2_{L^1(U_j(B))} \int_{r_B}^{2^{i+j-2}r_B} t^{2\epsilon-1}\,dt$$

$$\lesssim 2^{-2i(n+1+s)} 2^{-2j\varepsilon} \|\chi_B\|^{-2}_{L^\varphi(\mathbb{R}^n)}.$$

Moreover, when $t \in [2^{i+j-2}r_B, \infty)$, we see that, for all $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| = s+1$,

$$\int_{2^{i+j-2}r_B}^{\infty} \int_{B(x,t)} \left\{\int_{\mathbb{R}^n} \frac{1}{t^n} \left|(\partial_x^\gamma \phi)\left(\frac{\theta(y-z)+(1-\theta)(y-x_B)}{t}\right)\right|\right.$$

$$\times \left|\frac{z-x_B}{t}\right|^{s+1} \left|\left(\alpha\chi_{U_j(B)}\right)(z)\right| \,dz\Bigg\}^2 \frac{dy\,dt}{t^{n+1}}$$

$$\lesssim (2^j r_B)^{2(s+1)} \|\alpha\|^2_{L^1(U_j(B))} \int_{2^{i+j-2}r_B}^{\infty} t^{-2(n+s+1)-1}\,dt$$

$$\lesssim (2^j r_B)^{2(s+1)} (2^{i+j-2}r_B)^{-2(n+s+1)} \|\alpha\|^2_{L^q(U_j(B))} |U_j(B)|^{2/q'}$$

$$\lesssim 2^{-2i(n+s+1)} 2^{-2j\varepsilon} \|\chi_B\|^{-2}_{L^\varphi(\mathbb{R}^n)},$$

which, together with (4.19), implies that, for all $x \in U_i(B_j)$,

$$\mathrm{F}_j(x) \lesssim 2^{-i(n+s+1)} 2^{-j\varepsilon} \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}. \tag{4.20}$$

Then from (4.20), the uniformly lower type p_0 property of φ and Lemma 2.4(vi), it follows that, for each $i \in \mathbb{N}$ with $i \geq 4$ and $j \in \mathbb{Z}_+$,

$$\begin{aligned} \mathrm{F}_{i,j} &\lesssim \int_{U_i(B_j)} \varphi\left(x, 2^{-i(n+s+1)} 2^{-j\varepsilon} |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right) dx \\ &\lesssim 2^{-i(n+s+1)p_0} 2^{-j\varepsilon p_0} \varphi\left(2^{i+j}B, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right) \\ &\lesssim 2^{-ip_0(n+s+1-nq_0/p_0)} 2^{-jp_0(\varepsilon-nq_0/p_0)} \varphi\left(B, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right). \end{aligned} \tag{4.21}$$

Thus, by (4.9), (4.12), (4.17), (4.18), (4.21), $\epsilon > n(q_0/p_0 - 1)$ and $n+1+s > nq_0/p_0$, we conclude that

$$\int_{\mathbb{R}^n} \varphi\left(x, |\lambda| S(\alpha)(x)\right) \,dx \lesssim \varphi\left(B, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right),$$

which implies that (4.7) holds, and hence completes the proof of Proposition 4.7. $\square$

Recall that $f \in \mathcal{S}'(\mathbb{R}^n)$ is called to *vanish weakly at infinity*, if for every $\psi \in \mathcal{S}(\mathbb{R}^n)$, $f * \psi_t \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $t \to \infty$ (see, for example, [20, p. 50]). Then we have the following proposition for $H_{\varphi,S}(\mathbb{R}^n)$.

Proposition 4.8. *Let φ be as in Definition 2.2, $q \in (1,\infty)$, s as in Definition 4.1 and $\epsilon \in (nq(\varphi)/i(\varphi),\infty)$, where $q(\varphi)$ and $i(\varphi)$ are respectively as in (2.3) and (2.1). Assume that $f \in H_{\varphi,S}(\mathbb{R}^n)$ vanishes weakly at infinity. Then there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{\alpha_j\}_j$ of $(\varphi,\, q,\, s,\, \epsilon)$-molecules such that $f = \sum_j \lambda_j \alpha_j$ in $H_{\varphi,S}(\mathbb{R}^n)$. Moreover, there exists a positive constant C, independent of f, such that,*

$$\Lambda(\{\lambda_j\alpha_j\}_j) := \inf\left\{\lambda \in (0,\infty):\ \sum_j \varphi\left(B_j, \frac{|\lambda_j|}{\lambda\|\chi_{B_j}\|_{L^\varphi(\mathbb{R}^n)}}\right) \le 1\right\} \le C\|f\|_{H_{\varphi,S}(\mathbb{R}^n)},$$

where, for each j, α_j associates with the ball B_j.

Proof. By the assumptions of ϕ in Definition 4.2 and $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishing weakly at infinity, similar to the proof of [20, Theorem 1.64], we know that,

$$f = \int_0^\infty \phi_t * \phi_t * f\,\frac{dt}{t} \tag{4.22}$$

in $\mathcal{S}'(\mathbb{R}^n)$. Thus, $f = \pi_\phi(\phi_t * f)$ in $\mathcal{S}'(\mathbb{R}^n)$. Moreover, from $f \in H_{\varphi,S}(\mathbb{R}^n)$, we infer that $\phi_t * f \in T_\varphi(\mathbb{R}^{n+1}_+)$. Applying Theorem 3.1, Corollary 3.4 and Proposition 4.7(ii) to $\phi_t * f$, we conclude that

$$f = \pi_\phi(\phi_t * f) = \sum_j \lambda_j \pi_\phi(a_j) =: \sum_j \lambda_j \alpha_j$$

in both $\mathcal{S}'(\mathbb{R}^n)$ and $H_{\varphi,S}(\mathbb{R}^n)$, and $\Lambda(\{\lambda_j\alpha_j\}_j) \lesssim \|\phi_t * f\|_{T_\varphi(\mathbb{R}^{n+1}_+)} \sim \|f\|_{H_{\varphi,S}(\mathbb{R}^n)}$. Furthermore, by the proof of Proposition 4.7, we know that, for each j, α_j is a $(\varphi, q, s, \epsilon)$-molecule up to a harmless constant, which completes the proof of Proposition 4.8. $\square$

To establish the molecular and the Lusin area function characterization of $H_\varphi(\mathbb{R}^n)$, we need the atomic characterization of $H_\varphi(\mathbb{R}^n)$ obtained by Ky [34]. We begin with some notions.

Definition 4.9. Let φ be as in Definition 2.2.

(I) For each ball $B \subset \mathbb{R}^n$, the *space* $L^q_\varphi(B)$ with $q \in [1,\infty]$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ supported in B such that

$$\|f\|_{L^q_\varphi(B)} := \begin{cases} \sup\limits_{t\in(0,\infty)} \left[\dfrac{1}{\varphi(B,t)}\displaystyle\int_{\mathbb{R}^n} |f(x)|^q \varphi(x,t)\,dx\right]^{1/q} < \infty, & q \in [1,\infty), \\ \|f\|_{L^\infty(B)} < \infty, & q = \infty. \end{cases}$$

(II) A triplet $(\varphi,\, q,\, s)$ is called *admissible*, if $q \in (q(\varphi),\infty]$ and $s \in \mathbb{Z}_+$ satisfies $s \ge \lfloor n[\frac{q(\varphi)}{i(\varphi)} - 1]\rfloor$. A measurable function a on $\mathbb{R}^n$ is called a $(\varphi,\, q,\, s)$-*atom* if there exists a ball $B \subset \mathbb{R}^n$ such that

(i) $\operatorname{supp} a \subset B$;
(ii) $\|a\|_{L^q_\varphi(B)} \le \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}$;
(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \le s$.

(III) The *atomic Musielak-Orlicz Hardy space*, $H^{\varphi,\,q,\,s}(\mathbb{R}^n)$, is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that $f = \sum_j b_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{b_j\}_j$ is a sequence of multiples of $(\varphi,\,q,\,s)$-atoms with $\operatorname{supp} b_j \subset B_j$ and

$$\sum_j \varphi\left(B_j, \|b_j\|_{L^q_\varphi(B_j)}\right) < \infty.$$

Moreover, letting

$$\Lambda_q(\{b_j\}_j) := \inf\left\{\lambda \in (0,\infty):\ \ \sum_j \varphi\left(B_j, \frac{\|b_j\|_{L^q_\varphi(B_j)}}{\lambda}\right) \le 1\right\},$$

the *quasi-norm* of $f \in H^{\varphi,\,q,\,s}(\mathbb{R}^n)$ is defined by $\|f\|_{H^{\varphi,\,q,\,s}(\mathbb{R}^n)} := \inf\{\Lambda_q(\{b_j\}_j)\}$, where the infimum is taken over all the decompositions of f as above.

The following Lemma 4.10 is just [34, Theorem 3.1].

Lemma 4.10. *Let φ be as in Definition 2.2 and $(\varphi,\,q,\,s)$ admissible. Then $H_\varphi(\mathbb{R}^n) = H^{\varphi,\,q,\,s}(\mathbb{R}^n)$ with equivalent norms.*

Now we state the main theorem of this section as follows.

Theorem 4.11. *Let φ be as in Definition 2.2. Assume that $s \in \mathbb{Z}_+$ is as in Definition 4.2, $\varepsilon \in (\max\{n+s, nq(\varphi)/i(\varphi)\},\infty)$, $q \in (\max\{1/i(\varphi), q(\varphi)r(\varphi)/(r(\varphi)-1)\},\infty)$, where $q(\varphi)$, $i(\varphi)$ and $r(\varphi)$ are, respectively, as in* (2.3), (2.1) *and* (2.4). *Then the followings are equivalent:*
(i) *$f \in H_\varphi(\mathbb{R}^n)$;*
(ii) *$f \in H^{q,s,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$;*
(iii) *$f \in H_{\varphi,S}(\mathbb{R}^n)$ and f vanishes weakly at infinity.*

Moreover, for all $f \in H_\varphi(\mathbb{R}^n)$, $\|f\|_{H_\varphi(\mathbb{R}^n)} \sim \|f\|_{H^{q,s,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)} \sim \|f\|_{H_{\varphi,S}(\mathbb{R}^n)}$, where the implicit positive constants are independent of f.

To prove Theorem 4.11, we need the following Lemma 4.12.

Lemma 4.12. *Let φ be as in Definition 2.2. If $f \in H_\varphi(\mathbb{R}^n)$, then f vanishes weakly at infinity.*

Proof. Observe that for any $f \in H_\varphi(\mathbb{R}^n)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $t \in (0,\infty)$ and $y \in B(x,t)$, $|f*\phi_t(x)| \lesssim f^*(y)$, where f^* is as in Definition 4.1. Hence, since, for any $p \in (0, i(\varphi))$, φ is of uniformly lower type p, then by the uniformly lower type p and upper type 1 properties of φ and Lemma 2.3(iii), we conclude that, for all $x \in \mathbb{R}^n$,

$$\min\{|f*\phi_t(x)|^p, |f*\phi_t(x)|\} \lesssim [\varphi(B(x,t),1)]^{-1}\int_{B(x,t)} \varphi(y,1)\min\{[f^*(y)]^p, f^*(y)\}\,dy$$

$$\lesssim [\varphi(B(x,t),1)]^{-1} \int_{B(x,t)} \varphi(y, f^*(y))\, dy$$
$$\lesssim [\varphi(B(x,t),1)]^{-1} \max\{\|f\|^p_{H_\varphi(\mathbb{R}^n)}, \|f\|_{H_\varphi(\mathbb{R}^n)}\} \to 0,$$

as $t \to \infty$. That is, f vanishes weakly at infinity, which completes the proof of Lemma 4.12. □

Now we prove Theorem 4.11 by using Proposition 4.8, Lemmas 4.10 and 4.12.

Proof of Theorem 4.11. The proof of Theorem 4.11 is divided into the following three steps.

Step I. (i) $\Rightarrow$ (ii).

By Lemma 4.10, we see that $H_\varphi(\mathbb{R}^n) = H^{\varphi,\infty,s}(\mathbb{R}^n)$. Moreover, from the definitions of $H^{q,s,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$ and $H^{\varphi,\infty,s}(\mathbb{R}^n)$, we infer that $H^{\varphi,\infty,s}(\mathbb{R}^n) \subset H^{q,s,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$. Thus, $H_\varphi(\mathbb{R}^n) \subset H^{q,s,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$, which completes the proof of Step I.

Step II. (ii) $\Rightarrow$ (i).

For any fixed $(\varphi, q, s, \varepsilon)$-molecule α associated with the ball $B := B(x_B, r_B)$ and all $k \in \mathbb{Z}_+$, let $\alpha_k := \alpha\chi_{U_k(B)}$ and $\mathcal{P}_k$ be the *linear vector space generated by the set* $\{x^\alpha\chi_{U_k(B)}\}_{|\alpha|\le s}$ *of polynomials*. It is well known (see, for example, [53]) that there exists a unique *polynomial* $P_k \in \mathcal{P}_k$ such that for all multi-indices β with $|\beta| \le s$,

$$\int_{\mathbb{R}^n} x^\beta [\alpha_k(x) - P_k(x)]\, dx = 0, \tag{4.23}$$

where P_k is given by the following formula

$$P_k := \sum_{\beta\in\mathbb{Z}^n_+,\, |\beta|\le s} \left\{\frac{1}{|U_k(B)|}\int_{\mathbb{R}^n} x^\beta\, \alpha_k(x)\, dx\right\} Q_{\beta,k} \tag{4.24}$$

and $Q_{\beta,k}$ is the unique *polynomial* in $\mathcal{P}_k$ satisfying that, for all multi-indices β with $|\beta| \le s$ and the *dirac function* $\delta_{\gamma,\beta}$,

$$\int_{\mathbb{R}^n} x^\gamma\, Q_{\beta,k}(x)\, dx = |U_k(B)|\, \delta_{\gamma,\beta}. \tag{4.25}$$

By the assumption $q > q(\varphi)r(\varphi)/(r(\varphi)-1)$, we know that there exists $\widetilde{q} \in (q(\varphi),\infty)$ such that $q > \widetilde{q}r(\varphi)/(r(\varphi)-1)$ and hence $\varphi \in \mathbb{RH}_{(\frac{q}{\widetilde{q}})'}(\mathbb{R}^n)$. Now we prove that, for each $k \in \mathbb{Z}_+$, $\alpha_k - P_k$ is a $(\varphi, \widetilde{q}, s)$-atom, and $\sum_{k\in\mathbb{Z}_+} P_k$ can be divided into a sum of (φ, ∞, s)-atoms.

It was proved in [53] that, for all $k \in \mathbb{Z}_+$,

$$\sup_{x\in U_k(B)} |P_k(x)| \lesssim \frac{1}{|U_k(B)|}\|\alpha_k\|_{L^1(\mathbb{R}^n)},$$

which, together with Minkowski's inequality and Hölder's inequality, implies that

$$\|\alpha_k - P_k\|_{L^q(\mathbb{R}^n)} \lesssim \|\alpha_k\|_{L^q(2^kB)} + \|P_k\|_{L^q(2^kB)} \lesssim \|\alpha_k\|_{L^q(U_k(B))} \tag{4.26}$$

$$\lesssim 2^{-k\varepsilon}|2^kB|^{1/q}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}.$$

From this, Hölder's inequality and $\varphi\in\mathbb{RH}_{(\frac{q}{\widetilde{q}})'}(\mathbb{R}^n)$, it follows that

$$\begin{aligned}&\left\{\frac{1}{\varphi(2^kB,t)}\int_{2^kB}|\alpha_k(x)-P_k(x)|^{\widetilde{q}}\varphi(x,t)\,dx\right\}^{1/\widetilde{q}}\\&\quad\lesssim\frac{1}{[\varphi(2^kB,t)]^{1/\widetilde{q}}}\|\alpha_k-P_k\|_{L^q(2^kB)}\left\{\int_{2^kB}[\varphi(x,t)]^{(\frac{q}{\widetilde{q}})'}\,dx\right\}^{\frac{1}{\widetilde{q}(\frac{q}{\widetilde{q}})'}}\lesssim 2^{-k\varepsilon}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)},\end{aligned}$$

which implies that

$$\|\alpha_k-P_k\|_{L^\varphi_{\widetilde{q}}(2^kB)}\lesssim 2^{-k\varepsilon}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}.\tag{4.27}$$

This, combined with (4.23) and the fact that $\operatorname{supp}(\alpha_k-P_k)\subset 2^kB$, implies that for each $k\in\mathbb{Z}_+$, α_k-P_k is a multiple of a $(\varphi,\,\widetilde{q},\,s)$-atom.

Moreover, for any $j\in\mathbb{Z}_+$ and $\ell\in\mathbb{Z}_+^n$, let

$$N_\ell^j:=\sum_{k=j}^\infty|U_k(B)|\langle\alpha_k,x^\ell\rangle:=\sum_{k=j}^\infty\int_{U_k(B)}\alpha_k(x)x^\ell\,dx.$$

Then for any $\ell\in\mathbb{Z}_+^n$ with $|\ell|\le s$,

$$N_\ell^0=\sum_{k=0}^\infty\int_{U_k(B)}\alpha(x)x^\ell\,dx=0.$$

Therefore, by Hölder's inequality and the assumption $\varepsilon\in(n+s,\infty)$, we see that, for all $j\in\mathbb{Z}_+$ and $\ell\in\mathbb{Z}_+^n$ with $|\ell|\le s$,

$$\begin{aligned}|N_\ell^j|&\le\sum_{k=j}^\infty\int_{U_k(B)}|\alpha_j(x)x^\ell|\,dx\le\sum_{k=j}^\infty(2^kr_B)^{|\ell|}|2^kB|^{1/q'}\|\alpha_k\|_{L^q(U_j(B))}\\&\le\sum_{k=j}^\infty 2^{-k(\varepsilon-n-|\ell|)}|B|^{1+|\ell|/n}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\lesssim 2^{-j(\varepsilon-n-|\ell|)}|B|^{1+|\ell|/n}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}.\end{aligned}\tag{4.28}$$

Furthermore, from (4.25) and the homogeneity, we deduce that, for all $j\in\mathbb{Z}_+$, $\beta\in\mathbb{Z}_+^n$ with $|\beta|\le s$ and $x\in\mathbb{R}^n$, $|Q_{\beta,\,j}(x)|\lesssim\left(2^jr_B\right)^{-|\beta|}$, which, combining with (4.28), implies that, for all $j\in\mathbb{Z}_+$, $\ell\in\mathbb{Z}_+^n$ with $|\ell|\le s$ and $x\in\mathbb{R}^n$,

$$|U_j(B)|^{-1}\left|N_\ell^jQ_{\ell,j}(x)\chi_{U_j(B)}(x)\right|\lesssim 2^{-j\varepsilon}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}.\tag{4.29}$$

Moreover, by (4.24) and the definition of N_ℓ^j, we know that

$$\sum_{k=0}^\infty P_k=\sum_{\ell\in\mathbb{Z}_+^n,|\ell|\le s}\ \sum_{k=0}^\infty\sum_{j=1}^k\langle\alpha_j,x^\ell\rangle|U_j(B)|$$

$$= \sum_{\ell\in\mathbb{Z}_+^n,|\ell|\le s}\sum_{k=0}^{\infty} N_\ell^{k+1}\left[|U_k(B)|^{-1}Q_{\ell,k}\chi_{U_k(B)} - |U_{k+1}(B)|^{-1}Q_{\ell,k+1}\chi_{U_{k+1}(B)}\right]$$

$$=: \sum_{\ell\in\mathbb{Z}_+^n,|\ell|\le s}\sum_{k=0}^{\infty} b_\ell^k.$$

From (4.29), it follows that, for all $k\in\mathbb{Z}_+$ and $\ell\in\mathbb{Z}_+^n$ with $|\ell|\le s$,

$$\|b_\ell^k\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{-j\varepsilon}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}. \tag{4.30}$$

Moreover, by (4.25), we see that, for all $\gamma\in\mathbb{Z}_+^n$ with $|\gamma|\le s$, $\int_{\mathbb{R}^n} b_\ell^k(x)x^\gamma\,dx = 0$. Obviously, $\operatorname{supp}(b_\ell^k)\subset 2^{k+1}B$. Thus, b_ℓ^k is a multiple of a (φ,∞,s)-atom, and hence a multiple of a $(\varphi,\widetilde{q},s)$-atom. Furthermore, from the assumption $\varepsilon\in(nq(\varphi)/i(\varphi),\infty)$, we infer that there exist $p_0\in(0,i(\varphi))$ and $q_0\in(q(\varphi),\infty)$ such that $\varepsilon>nq_0/p_0$. Then $\varphi\in\mathbb{A}_{q_0}(\mathbb{R}^n)$ and φ is of uniformly lower type p_0. By (4.27), (4.30), the uniformly lower type p_0 property of φ and $\varepsilon>nq_0/p_0$, we conclude that, for all $\lambda\in(0,\infty)$,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}_+}\varphi\left(2^kB,\lambda\|\alpha_k-P_k\|_{L_\varphi^{\widetilde{q}}(2^kB)}\right)+\sum_{|\ell|\le s}\sum_{k\in\mathbb{Z}_+}\varphi\left(2^{k+1}B,\lambda\|b_\ell^k\|_{L_\varphi^\infty(2^{k+1}B)}\right)\\
&\quad\lesssim\sum_{k\in\mathbb{Z}_+}2^{-p_0k\varepsilon}\varphi\left(2^{k+1}B,\lambda\|\chi_B\|_{L^\varphi(B)}^{-1}\right)\\
&\quad\lesssim\sum_{k\in\mathbb{Z}_+}2^{-p_0k(\varepsilon-nq_0/p_0)}\varphi\left(B,\lambda\|\chi_B\|_{L^\varphi(B)}^{-1}\right)\lesssim\varphi\left(B,\lambda\|\chi_B\|_{L^\varphi(B)}^{-1}\right).
\end{aligned}\tag{4.31}$$

Let $f\in H_{\varphi,\mathrm{mol}}^{q,s,\varepsilon}(\mathbb{R}^n)$. Then there exist $\{\lambda_j\}_j\subset\mathbb{C}$ and a sequence $\{\alpha_j\}_j$ of $(\varphi,q,s,\varepsilon)$-molecules such that $f=\sum_j\lambda_j\alpha_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\|f\|_{H_{\varphi,\mathrm{mol}}^{q,s,\varepsilon}(\mathbb{R}^n)}\sim\Lambda(\{\lambda_j\alpha_j\}_j). \tag{4.32}$$

Let $p_1\in(1,q)$. Then by (4.26) and (4.30), we know that for each j, there exist a sequence $\{a_{j,k}\}_k$ of multiples of $(\varphi,\widetilde{q},s)$-atoms such that $\alpha_j=\sum_k a_{j,k}$ in $L^{p_1}(\mathbb{R}^n)$. Thus, $f=\sum_j\sum_k\lambda_j a_{j,k}$ in $\mathcal{S}'(\mathbb{R}^n)$, which, together with Lemma 4.10, implies that $f\in H_\varphi(\mathbb{R}^n)$. Moreover, from (4.31) and (4.32), it follows that

$$\|f\|_{H_\varphi(\mathbb{R}^n)}\lesssim\Lambda(\{\lambda_j a_{j,k}\}_{j,k})\lesssim\Lambda(\{\lambda_j\alpha_j\}_j)\sim\|f\|_{H_{\varphi,\mathrm{mol}}^{q,s,\varepsilon}(\mathbb{R}^n)},$$

which completes the proof of Step II.

Step III. (ii) $\Leftrightarrow$ (iii).

Let $f\in H_{\varphi,S}(\mathbb{R}^n)$ vanishing weakly at infinity. Then from Proposition 4.8, it follows that $f\in H_{\varphi,\mathrm{mol}}^{q,s,\varepsilon}(\mathbb{R}^n)$ and $\|f\|_{H_{\varphi,\mathrm{mol}}^{q,s,\varepsilon}(\mathbb{R}^n)}\lesssim\|f\|_{H_{\varphi,S}(\mathbb{R}^n)}$.

Conversely, assume that $f\in H_{\varphi,\mathrm{mol}}^{q,s,\varepsilon}(\mathbb{R}^n)$. Then by Steps I and II, we know that $H_{\varphi,\mathrm{mol}}^{q,s,\varepsilon}(\mathbb{R}^n)=H_\varphi(\mathbb{R}^n)$, which, together with Lemma 4.12, implies that f vanishes weakly at infinity. Moreover, from (4.7), together with a standard argument, we infer that $f\in H_{\varphi,S}(\mathbb{R}^n)$. This finishes the proof of Step III and hence Theorem 4.11. $\square$

Remark 4.13. By Theorem 4.11, we see that the Hardy-type space $H_{\varphi,S}(\mathbb{R}^n)$ is independent of the choices of ϕ as in Definition 4.2, and the Hardy-type space $H^{q,s,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$ is independent of the choices of q, s and ε as in Theorem 4.11.

5 The Carleson measure characterization of $\mathrm{BMO}_\varphi(\mathbb{R}^n)$

In this section, we first recall the notion of the Musielak-Orlicz BMO-type space $\mathrm{BMO}_\varphi(\mathbb{R}^n)$ from [34] and introduce the φ-Carleson measure. Then we establish the φ-Carleson measure characterization of $\mathrm{BMO}_\varphi(\mathbb{R}^n)$ by using the Lusin area function characterization of $H_\varphi(\mathbb{R}^n)$ obtained in Theorem 4.11.

The following Musielak-Orlicz BMO-type space $\mathrm{BMO}_\varphi(\mathbb{R}^n)$ was introduced by Ky [34].

Definition 5.1. Let φ be as in Definition 2.2. A locally integrable function f on $\mathbb{R}^n$ is said to belong to the *space* $\mathrm{BMO}_\varphi(\mathbb{R}^n)$, if

$$\|f\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \int_B |f(x) - f_B|\, dx < \infty,$$

where, and in what follows, the supremum is taken over all the balls $B \subset \mathbb{R}^n$ and

$$f_B := \frac{1}{|B|} \int_B f(y)\, dy.$$

Definition 5.2. Let φ be as in Definition 2.2. A measure $d\mu$ on $\mathbb{R}^{n+1}_+$ is called a φ-*Carleson measure* if

$$\|d\mu\|_\varphi := \sup_{B\subset\mathbb{R}^n} \frac{|B|^{1/2}}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \left\{\int_{\widehat{B}} |d\mu(x,t)|\right\}^{1/2} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and $\widehat{B}$ denotes the tent over B.

Theorem 5.3. *Let φ be as in Definition 2.2 and ϕ as in Definition 4.2.*

(i) *Assume that $b \in \mathrm{BMO}_\varphi(\mathbb{R}^n)$. Then $d\mu(x,t) := |\phi_t * b(x)|^2 \frac{dx\,dt}{t}$ is a φ-Carleson measure on $\mathbb{R}^{n+1}_+$; moreover, there exists a positive constant C, independent of b, such that $\|d\mu\|_\varphi \le C\|b\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)}$.*

(ii) *Assume further that $nq(\varphi) < (n+1)i(\varphi)$. Let $b \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and, for all $(x,t) \in \mathbb{R}^{n+1}_+$,*

$$d\mu(x,t) := |\phi_t * b(x)| \frac{dxdt}{t}$$

be a φ-Carleson measure on $\mathbb{R}^{n+1}_+$. Then $b \in \mathrm{BMO}_\varphi(\mathbb{R}^n)$ and, moreover, there exists a positive constant C, independent of b, such that $\|b\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)} \le C\|d\mu\|_\varphi$.

To prove Theorem 5.3, we need the following several lemmas.

Lemma 5.4. *Let φ be as in Definition 2.2 and $f \in \mathrm{BMO}_\varphi(\mathbb{R}^n)$. Then there exist positive constants C_1 and C_2, independent of f, such that for all balls $B \subset \mathbb{R}^n$ and $\lambda \in (0,\infty)$,*

$$|\{x \in B: \ |f(x) - f_B| > \lambda\}| \le C_1 \exp\left\{-\frac{C_2|B|\lambda}{\|f\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right\}|B|.$$

Proof. Let $f \in \mathrm{BMO}_\varphi(\mathbb{R}^n)$. Take the ball $B_0 \subset \mathbb{R}^n$. By dilation and translation, without loss of generality, we may assume that $\|f\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)}\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)} = |B_0|$ and $f_{B_0} = 0$; otherwise, we replace f by $\frac{(f-f_{B_0})|B_0|}{\|f\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)}\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)}}$. Thus, we only need to prove that there exist positive constants C and c, independent of f and B_0, such that for any $\lambda \in (0,\infty)$, $|\{x \in B_0 : \ |f(x)| > \lambda\}| \le Ce^{-c\lambda}|B_0|$, whose proof is standard and we omit the details (see, for example, [32]). This finishes the proof of Lemma 5.4. $\square$

By Hölder's inequality and Lemma 5.4, we obtain the following Corollary 5.5 immediately. We omit the details.

Corollary 5.5. *Let $p \in (1,\infty)$ and φ be as in Definition 2.2. Then $f \in \mathrm{BMO}_\varphi(\mathbb{R}^n)$ if and only if $f \in \mathrm{BMO}^p_\varphi(\mathbb{R}^n)$, where*

$$\mathrm{BMO}^p_\varphi(\mathbb{R}^n) := \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \ \|f\|_{\mathrm{BMO}^p_\varphi(\mathbb{R}^n)} < \infty\}$$

and

$$\|f\|_{\mathrm{BMO}^p_\varphi(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{|B|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \left\{\frac{1}{|B|}\int_B |f(x) - f_B|^p \, dx\right\}^{1/p}$$

with the supremum taken over all the balls $B \subset \mathbb{R}^n$ and $f_B := \frac{1}{|B|}\int_B f(y)\, dy$.

Lemma 5.6. *Let φ be as in Theorem 5.3, $\epsilon \in (n[\frac{q(\varphi)}{i(\varphi)} - 1], \infty)$ with $q(\varphi)$ and $i(\varphi)$ being respectively as in (2.3) and (2.1), and $B_0 := B(x_0, \delta)$. Then there exists a positive constant C such that for all $f \in \mathrm{BMO}_\varphi(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \frac{\delta^\epsilon |f(x) - f_{B_0}|}{\delta^{n+\epsilon} + |x - x_0|^{n+\epsilon}}\, dx \le C\frac{\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)}}{|B_0|}\|f\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)}.$$

Proof. For any $k \in \mathbb{Z}_+$, let $B_k := 2^k B_0$. Then for all $k \in \mathbb{Z}_+$,

$$|f_{2^{k+1}B} - f_{2^kB}| \le \frac{2^n}{|B_{k+1}|}\int_{B_{k+1}} |f(x) - f_{B_{k+1}}|\, dx \le 2^n \frac{\|\chi_{B_{k+1}}\|_{L^\varphi(\mathbb{R}^n)}}{|B_{k+1}|}\|f\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)}. \tag{5.1}$$

By $\epsilon \in (n[\frac{q(\varphi)}{i(\varphi)} - 1], \infty)$, we know that there exist $p_0 \in (0, i(\varphi))$ and $q_0 \in (q(\varphi), \infty)$ such that $\epsilon > n(\frac{q_0}{p_0} - 1)$. Then $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$ and φ is of uniformly lower type p_0, which, together with Lemma 2.4(vi), implies that, for all $j \in \mathbb{Z}_+$,

$$\varphi\left(B_j, 2^{-jnq_0/p_0}\|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right) \lesssim 2^{-jnq_0}\varphi\left(B_j, \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right) \lesssim 1.$$

From this, we deduce that, for all $j \in \mathbb{Z}_+$, $\|\chi_{B_j}\|_{L^\varphi(\mathbb{R}^n)} \lesssim 2^{jnq_0/p_0}\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)}$, which, together with (5.1), implies that for all $k \in \mathbb{N}$,

$$|f_{B_k} - f_{B_0}| \le 2^n \|f\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)} \left(\frac{\|\chi_{B_k}\|_{L^\varphi(\mathbb{R}^n)}}{|B_k|} + \cdots + \frac{\|\chi_{2B_0}\|_{L^\varphi(\mathbb{R}^n)}}{|2B_0|}\right)$$

$$\lesssim \left\{\sum_{j=1}^{k} 2^{jn(q_0/p_0-1)}\right\} \frac{\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)}}{|B_0|}\|f\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)}$$
$$\lesssim 2^{kn(q_0/p_0-1)} \frac{\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)}}{|B_0|}\|f\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)}.$$

By this, we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\delta^\epsilon|f(x)-f_{B_0}|}{\delta^{n+\epsilon}+|x-x_0|^{n+\epsilon}}\,dx \\
&\quad\le \int_{B_0} \frac{\delta^\epsilon|f(x)-f_{B_0}|}{\delta^{n+\epsilon}+|x-x_0|^{n+\epsilon}}\,dx + \sum_{k=0}^{\infty}\int_{B_{k+1}\setminus B_k}\cdots \\
&\quad\lesssim \int_{B_0} \frac{\delta^\epsilon|f(x)-f_{B_0}|}{\delta^{n+\epsilon}}\,dx + \sum_{k=1}^{\infty}(2^k\delta)^{-(n+\epsilon)}\delta^\epsilon\int_{B_k}|f(x)-f_{B_0}|\,dx \\
&\quad\lesssim \frac{\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)}}{|B_0|}\|f\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)} + \sum_{k=1}^{\infty}2^{-k(n+\epsilon)}\delta^{-n}\left[\int_{B_k}|f(x)-f_{B_k}|\,dx+|f_{B_k}-f_{B_0}|\right] \\
&\quad\lesssim \left\{\sum_{k=1}^{\infty}2^{-k(n+\epsilon-nq_0/p_0)}\right\}\frac{\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)}}{|B_0|}\|f\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)} \lesssim \frac{\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)}}{|B_0|}\|f\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)},
\end{aligned}$$

which completes the proof of Lemma 5.6. □

Denote by $H^{\varphi,\infty,s}_{\mathrm{fin}}(\mathbb{R}^n)$ the *sets of all finite combinations of* (φ,∞,s)*-atoms*. It is easy to see that $H^{\varphi,\infty,s}_{\mathrm{fin}}(\mathbb{R}^n)$ is dense in $H^{\varphi,\infty,s}(\mathbb{R}^n)$. The following Lemma 5.7 is just [34, Theorem 3.2].

Lemma 5.7. *Let φ be as in Definition 2.2 satisfying $nq(\varphi)<(n+1)i(\varphi)$, where $q(\varphi)$ and $i(\varphi)$ are respectively as in (2.3) and (2.1). Then the dual space of $H_\varphi(\mathbb{R}^n)$, denoted by $(H_\varphi(\mathbb{R}^n))^*$, is $\mathrm{BMO}_\varphi(\mathbb{R}^n)$ in the following sense:*

(i) *Suppose that $b\in\mathrm{BMO}_\varphi(\mathbb{R}^n)$. Then the linear functional $L_b:\ f\to L_b(f):=\int_{\mathbb{R}^n}f(x)b(x)\,dx$, initially defined for $H^{\varphi,\infty,s}_{\mathrm{fin}}(\mathbb{R}^n)$, has a bounded extension to $H_\varphi(\mathbb{R}^n)$.*

(ii) *Conversely, every continuous linear functional on $H_\varphi(\mathbb{R}^n)$ arises as the above with a unique $b\in\mathrm{BMO}_\varphi(\mathbb{R}^n)$.*

Moreover, $\|b\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)}\sim\|L_b\|_{(H_\varphi(\mathbb{R}^n))^*}$, where the implicit constants are independent of b.

Proof of Theorem 5.3. We first prove (i). For any given ball $B_0:=B(x_0,r_0)$, let $\widetilde{B}:=2B_0$. Then

$$b=b_{\widetilde{B}}+(b-b_{\widetilde{B}})\chi_{\widetilde{B}}+(b-b_{\widetilde{B}})\chi_{\mathbb{R}^n\setminus\widetilde{B}}=:b_1+b_2+b_3. \tag{5.2}$$

For b_1, by $\int_{\mathbb{R}^n}\phi(x)\,dx=0$, we see that, for all $t\in(0,\infty)$, $\phi_t*b_1\equiv0$, which implies that

$$\int_{\widehat{B_0}}|\phi_t*b_1(x)|^2\frac{dx\,dt}{t}=0. \tag{5.3}$$

For b_2, from Proposition 4.7(i), it follows that

$$\int_{\widehat{B_0}} |\phi_t * b_2(x)|^2 \frac{dx\,dt}{t} \le \int_{\mathbb{R}^{n+1}_+} |\phi_t * b_2(x)|^2 \frac{dx\,dt}{t} \lesssim \|b_2\|^2_{L^2(\mathbb{R}^n)} \sim \int_{\widetilde{B}} |b(x) - b_{\widetilde{B}}|^2\, dx,$$

which, together with Corollary 5.5, implies that

$$\begin{aligned}(5.4)\qquad & \frac{|B_0|}{\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)}} \left\{\frac{1}{|B_0|}\int_{\widehat{B_0}} |\phi_t * b_2(x)|^2 \frac{dx\,dt}{t}\right\}^{1/2} \\ &\quad \lesssim \frac{|B_0|}{\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)}} \left\{\frac{1}{|B_0|}\int_{\widetilde{B}} |b(x) - b_{\widetilde{B}}|^2\right\}^{1/2} \lesssim \|b\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)}.\end{aligned}$$

Let ϵ be as in Lemma 5.6. Furthermore, for any $(x,t) \in \widehat{B}_0$ and $y \in (\widetilde{B})^\complement$, we see that $t \in (0, r_{B_0})$ and $|y - x| \gtrsim |y - x_0|$. Then by Lemma 5.6, we conclude that

$$\begin{aligned}|\phi_t * b_3(x)| &\lesssim \int_{(\widetilde{B})^\complement} \frac{t^\epsilon |b(x) - b_{\widetilde{B}}|}{(t + |x-y|)^{n+\epsilon}}\, dy \\ &\lesssim \int_{(\widetilde{B})^\complement} \frac{t^\epsilon |b(x) - b_{\widetilde{B}}|}{|y - x_B|^{n+\epsilon}}\, dy \lesssim \frac{t^\epsilon}{r_B^\epsilon} \frac{\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)}}{|B_0|} \|b\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)},\end{aligned}$$

which implies that

$$\begin{aligned}&\frac{|B_0|}{\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)}} \left\{\frac{1}{|B_0|}\int_{\widehat{B_0}} |\phi_t * b_3(x)|^2 \frac{dx\,dt}{t}\right\}^{1/2} \\ &\quad \lesssim \left\{\int_0^{r_B} \frac{t^{2\epsilon-1}}{t_B^{2\epsilon}}\, dt\right\}^{1/2} \|b\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)}.\end{aligned}$$

From this, (5.2), (5.3) and (5.4), we deduce that

$$\frac{|B_0|}{\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)}} \left\{\frac{1}{|B_0|}\int_{\widehat{B_0}} |\phi_t * b(x)|^2 \frac{dx\,dt}{t}\right\}^{1/2} \lesssim \|b\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)},$$

which, together with the arbitrariness of $B_0 \subset \mathbb{R}^n$, implies that $d\mu$ is a φ-Carleson measure on $\mathbb{R}^{n+1}_+$ and $\|d\mu\|_\varphi \lesssim \|b\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)}$. This finishes the proof of (i).

Now we prove (ii). Let $f \in H^{\varphi,\infty,s}_{\mathrm{fin}}(\mathbb{R}^n)$. Then by $f \in L^\infty(\mathbb{R}^n)$ with compact support, $b \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and the Plancherel formula, we conclude that

$$(5.5)\qquad \int_{\mathbb{R}^n} f(x)\overline{b(x)}\, dx = \int_{\mathbb{R}^{n+1}_+} \phi_t * f(x) \overline{\phi_t * b(x)}\, \frac{dx\,dt}{t},$$

where $\overline{b(x)}$ and $\overline{\phi_t * b(x)}$ denote, respectively, the conjugates of $b(x)$ and $\phi_t * b(x)$. Moreover, from $f \in H^{\varphi,\infty,s}_{\mathrm{fin}}(\mathbb{R}^n)$ and Theorem 4.11, it follows that $f \in H_{\varphi,S}(\mathbb{R}^n)$, which further implies that $\phi_t * f \in T_\varphi(\mathbb{R}^{n+1}_+)$. By this and Theorem 3.1, we conclude that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{a_j\}_j$ of (φ,∞)-atoms such that $\phi_t * f = \sum_j \lambda_j a_j$. From this,

(5.5), Hölder's inequality, (3.2), Theorem 4.11 and the uniformly upper type 1 property of φ, we deduce that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)\overline{b(x)}\,dx\right| &\le \sum_j |\lambda_j| \int_{\mathbb{R}^{n+1}_+} |a_j(x,t)||\phi_t * b(x)|\,\frac{dx\,dt}{t}\\ &\le \sum_j |\lambda_j| \left\{\int_{\widehat{B}_j} |a_j(x,t)|^2 \frac{dx\,dt}{t}\right\}^{1/2} \left\{\int_{\widehat{B}_j} |\phi_t * b(x)|^2 \frac{dx\,dt}{t}\right\}^{1/2}\\ &\lesssim \sum_j |\lambda_j||B_j|^{1/2}\|\chi_{B_j}\|^{-1}_{L^\varphi(\mathbb{R}^n)} \left\{\int_{\widehat{B}_j} |\phi_t * b(x)|^2 \frac{dx\,dt}{t}\right\}^{1/2}\\ &\lesssim \sum_j |\lambda_j|\|d\mu\|_\varphi \lesssim \Lambda(\{\lambda_j a_j\}_j)\|d\mu\|_\varphi \lesssim \|f\|_{H_{\varphi,S}(\mathbb{R}^n)}\|d\mu\|_\varphi\\ &\lesssim \|f\|_{H_\varphi(\mathbb{R}^n)}\|d\mu\|_\varphi,\end{aligned}$$

which implies that $\|b\|_{\mathrm{BMO}_\varphi(\mathbb{R}^n)} \lesssim \|d\mu\|_\varphi$, and hence completes the proof Theorem 5.3. $\square$

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